A Complete Dichotomy Rises from the Capture of Vanishing Signatures

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Abstract

We prove a complexity dichotomy theorem for Holant problems over an arbitrary set of complex-valued symmetric constraint functions $\mathcal{F}$ on Boolean variables. This extends and unifies all previous dichotomies for Holant problems on symmetric constraint functions (taking values without a finite modulus). We define and characterize all symmetric *vanishing* signatures. They turned out to be essential to the complete classification of Holant problems. The dichotomy theorem has an explicit tractability criterion. A Holant problem defined by a set of constraint functions $\mathcal{F}$ is solvable in polynomial time if it satisfies this tractability criterion, and is #P-hard otherwise. The tractability criterion can be intuitively stated as follows: A set $\mathcal{F}$ is tractable if (1) every function in $\mathcal{F}$ has arity at most two, or (2) $\mathcal{F}$ is transformable to an affine type, or (3) $\mathcal{F}$ is transformable to a product type, or (4) $\mathcal{F}$ is vanishing, combined with the right type of binary functions, or (5) $\mathcal{F}$ belongs to a special category of vanishing type Fibonacci gates. The proof of this theorem utilizes many previous dichotomy theorems on Holant problems and Boolean #CSP. Holographic transformations play an indispensable role, not only as a proof technique, but also in the statement of the dichotomy criterion.

1 Introduction

In the study of counting problems, several interesting frameworks of increasing generality have been proposed. One is called H-coloring or Graph Homomorphism [38, 29, 23, 1, 22, 4, 26, 6]. Another is called Constraint Satisfaction Problems (#CSP) [3, 2, 1, 11, 7, 8, 24, 21, 27, 10, 5]. Recently, inspired by Valiant's holographic algorithms [44, 43], a further refined framework called Holant problems [16, 17, 11, 13] was proposed. They all describe classes of counting problems that can be expressed as a sum-of-product computation, specified by a set of local constraint functions $\mathcal{F}$, also called signatures. They differ mainly in what $\mathcal{F}$ can be and what is assumed to be present in $\mathcal{F}$ by default. Such frameworks are interesting because the language is *expressive* enough so that they contain many natural counting problems, while *specific* enough so that it is possible to prove *dichotomy theorems*. Such theorems completely classify every problem in a class to be either in P or #P-hard [40, 18, 25, 19].

The goal is to understand which counting problems are computable in polynomial time (called tractable) and which are not (called intractable). We aim for a characterization in terms of $\mathcal{F}$. An ideal outcome is to be able to classify, within a broad class of functions, *every* function set $\mathcal{F}$ according to whether it defines a tractable counting problem or a #P-hard one. We note that, by an analogue of Ladner's theorem [36], such a dichotomy is *false* for the whole of #P, unless P = #P.

We give a brief description of the Holant framework here [16, 17, 11, 13]. A *signature grid* $\Omega = (G, \mathcal{F}, \pi)$ is a tuple, where $G = (V, E)$ is a graph, π labels each $v \in V$ with a function $f_v \in \mathcal{F}$, and f_v maps $\{0,1\}^{\deg(v)}$ to $\mathbb{C}$. We consider all 0-1 edge assignments. An assignment σ for every $e \in E$ gives an evaluation $\prod_{v \in V} f_v(\sigma \mid_{E(v)})$, where $E(v)$ denotes the incident edges of v and $\sigma \mid_{E(v)}$ denotes the restriction of σ to $E(v)$. The counting problem on the instance Ω is to compute

$$\text{Holant}_\Omega = \sum_{\sigma: E \to \{0,1\}} \prod_{v \in V} f_v \left(\sigma \mid_{E(v)}\right). \tag{1}$$

For example, consider the problem of counting Perfect Matching on G. This problem corresponds to attaching the Exact-One function at every vertex of G.

The Holant framework can be defined for general domain $[q]$; in this paper we restrict to the Boolean case $q = 2$. The #CSP problems are the special case of Holant problems where all Equality functions (with any number of inputs) are assumed to be included in $\mathcal{F}$. Graph Homomorphism is the further special case of #CSP where $\mathcal{F}$ consists of a single binary function (in addition to all Equality functions). Similar or essentially the same notions as Holant have been studied as tensor networks [31, 39] in physics, as Forney graphs and sum-product algorithms of factor graphs [32, 37] in artificial intelligence, coding theory, and signal processing.

Consider the following constraint function $f : \{0,1\}^4 \to \mathbb{C}$. Let the input (x_1, x_2, x_3, x_4) have Hamming weight w, then $f(x_1, x_2, x_3, x_4) = 3, 0, 1, 0, 3$, if $w = 0, 1, 2, 3, 4$, respectively. We denote this function by $f = [3, 0, 1, 0, 3]$. What is the counting problem defined by the Holant sum in equation (1) on 4-regular graphs G when $\mathcal{F} = \{f\}$? By definition, this is a sum over all 0-1 edge assignments of products of local evaluations. We only sum over assignments which assign an even number of 1's to the incident edges of each vertex, since $f = 0$ for $w = 1$ and 3. Then each vertex contributes a factor 3 if the 4 incident edges are assigned all 0 or all 1, and contributes a factor 1 if exactly two incident edges are assigned 1. *Before anyone thinks that this problem is artificial*, let's consider a holographic transformation. Consider the edge-vertex incident graph $H = (E(G), V(G), \{(e, v) \mid v \text{ is incident to } e \text{ in } G\})$ of G. This Holant problem can be expressed in the bipartite form $\text{Holant}\,(=_2 \mid f)$ on H, where $=_2$ is the binary Equality function. Thus, every $e \in E(G)$ is assigned $=_2$, and every $v \in V(G)$ is assigned f. We can write $=_2$ by its truth table $(1, 0, 0, 1)$ indexed by $\{0,1\}^2$. If we apply the holographic transformation $Z = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]$, then Valiant's Holant Theorem [44] tells us that $\text{Holant}\,(=_2 \mid f)$ is exactly the same as $\text{Holant}\left((=_2)Z^{\otimes 2} \mid (Z^{-1})^{\otimes 4} f\right)$. Here $(=_2)Z^{\otimes 2}$ is a row vector indexed by $\{0,1\}^2$ denoting the transformed function under Z from $(=_2) = (1, 0, 0, 1)$, and $(Z^{-1})^{\otimes 4} f$ is the column vector indexed by $\{0,1\}^4$ denoting the transformed function under Z^{-1} from f. Let $\hat{f}$ be the Exact-Two function on $\{0,1\}^4$. We can write its truth table as a column vector indexed by $\{0,1\}^4$, which has a value 1 at Hamming weight two and 0 elsewhere. In symmetric signature notation, $\hat{f} = [0, 0, 1, 0, 0]$. Then we have

$$\begin{aligned}
Z^{\otimes 4} \hat{f} &= Z^{\otimes 4} \{ \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \\
&\qquad + \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \} \\
&= \tfrac{1}{4} \{ \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right] + \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right] + \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right] \\
&\qquad + \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right] + \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right] + \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right] \} \\
&= \tfrac{1}{2} [3, 0, 1, 0, 3] = \tfrac{1}{2} f;
\end{aligned}$$

hence $(Z^{-1})^{\otimes 4} f = 2\hat{f}$. (Here we use the elementary fact that $(A \otimes B)(u \otimes v) = Au \otimes Bv$ for tensor products of matrices and vectors.) Meanwhile, Z transforms $=_2$ to the binary Disequality function $\neq_2$:

$$(=_2)Z^{\otimes 2} = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} Z^{\otimes 2} = \left\{ \begin{pmatrix} 1 & 0 \end{pmatrix}^{\otimes 2} + \begin{pmatrix} 0 & 1 \end{pmatrix}^{\otimes 2} \right\} Z^{\otimes 2} = \frac{1}{2} \left\{ \begin{pmatrix} 1 & 1 \end{pmatrix}^{\otimes 2} + \begin{pmatrix} i & -i \end{pmatrix}^{\otimes 2} \right\} = [0, 1, 0] = (\neq_2).$$

Hence, up to a global constant factor of 2^n on a graph with n vertices, the Holant problem with $[3, 0, 1, 0, 3]$ is exactly the same as $\operatorname{Holant}(\neq_2 \mid [0, 0, 1, 0, 0])$. A moment's reflection shows that this latter problem is counting the number of Eulerian orientations on 4-regular graphs, an eminently natural problem! Thus holographic transformations can reveal the fact that completely different looking problems are really the same problem, and there is no objective criterion on one problem being more "natural" than another. Hence we would like to classify all Holant problems given by such signatures.

An interesting observation is that $\operatorname{Holant}(\neq_2 \mid [0, 0, 1, 0, 0])$ has exactly the same value as $\operatorname{Holant}(\neq_2 \mid [a, b, 1, 0, 0])$ on any signature grid, for any $a, b \in \mathbb{C}$. This is because on a bipartite graph, $\neq_2$ demands that exactly half of the edges are 0 and the other half are 1, while on the other side, any use of the value a or b results in strictly less than half of the edges being 1. This is related to a phenomenon we call *vanishing*. Vanishing signatures are constraint functions, that when applied to any signature grid, produce a zero Holant value. A simple example is a tensor product of $\begin{pmatrix} 1 & i \end{pmatrix}$, i.e., a constraint function of the form $\begin{pmatrix} 1 & i \end{pmatrix}^{\otimes k}$ on k variables. This function on a vertex (of degree k) can be replaced by k copies of the unary function $\begin{pmatrix} 1 & i \end{pmatrix}$ on k new vertices, each connected to an incident edge. Whenever two copies of $\begin{pmatrix} 1 & i \end{pmatrix}$ meet in the evaluation of Holant in equation (1), they annihilate each other since they give the value $\begin{pmatrix} 1 & i \end{pmatrix} \cdot \begin{pmatrix} 1 & i \end{pmatrix} = 0$. These ghostly constraint functions are like the elusive dark matter. They do not actually contribute any value to the Holant sum. However in order to give a complete dichotomy for Holant problems, it turns out to be essential that we capture these vanishing signatures. There is another similarity with dark matter. Their contribution to the Holant sum is not directly observed. Yet in terms of the dimension of the algebraic variety they constitute, they make up the vast majority of the tractable symmetric signatures. Furthermore, when combined with others, they provide a large substrate to produce non-vanishing and tractable signatures. In #CSP problems, they are invisible due to the presumed inclusion of all the EQUALITY functions; and they lurk beneath the surface when one only considers real-valued Holant problems.

The existence of vanishing signatures have influenced previous dichotomy results, although this influence was not fully recognized at the time. In the dichotomy theorems in [11] and in [8], almost all tractable signatures can be transformed into a tractable #CSP problem, except for one special category. The tractability proof for this category used the fact that they are a special case of generalized Fibonacci signatures [16]. However, what went completely unnoticed is that for every input instance using such signatures alone, the Holant value is always zero!

The most significant previous encounter with vanishing signatures was in the parity setting [28]. The authors noticed that a large fraction of signatures always induce an even Holant value, which is vanishing in $\mathbb{Z}_2$. However, the parity dichotomy was achieved using an existential argument without obtaining a complete characterization of the vanishing signatures. Consequently, the dichotomy criterion is non-constructive and is currently not known to be decidable. Nevertheless, this work is important because it was the first to discover nontrivial vanishing signatures in the parity setting and to obtain a dichotomy that was *completed* by vanishing signatures.

To complement our characterization of vanishing signatures, we also obtain a characterization of signatures *transformable* to the #CSP tractable *Affine* type $\mathscr{A}$ or *Product* type $\mathscr{P}$, after an orthogonal holographic transformation. An orthogonal transformation is natural since the binary

EQUALITY $=_2$ is unchanged under such holographic transformations. With explicit characterizations of these tractable signatures, a complete dichotomy theorem becomes possible.

We first prove a dichotomy for a single signature, and then we extend it to an arbitrary set of signatures. The most difficult part is to prove a dichotomy for a single signature of arity 4. The proof involves a demanding interpolation step and an approximation argument, both of which use asymmetric signatures. We found that in order to prove a dichotomy for symmetric signatures, we must go through asymmetric signatures.

With this dichotomy, we come to a conclusion on a long series of dichotomies on Holant problems [17, 11, 14, 34, 35, 9, 10, 8, 30]. They all become special cases of this dichotomy. However, the proof of this theorem is logically dependent on some of these previous dichotomies. In particular, this dichotomy extends the dichotomy in [30] that covers all real-valued symmetric signatures. While we do not rely on their real-valued dichotomy itself, we do make important use of two results in [30]. One is the #P-hardness of the Eulerian orientations problem; the other is a dichotomy for $\#\mathrm{CSP}^d$, where every variable appears a multiple of d times.

2 Preliminaries

2.1 Problems and Definitions

The framework of Holant problems is defined for functions mapping any $[q]^k \to \mathbb{F}$ for a finite q and some field $\mathbb{F}$. In this paper, we investigate the complex-weighted Boolean Holant problems, that is, all functions are $[2]^k \to \mathbb{C}$. Strictly speaking, for consideration of models of computation, functions take complex algebraic numbers.

A *signature grid* $\Omega = (G, \mathcal{F}, \pi)$ consists of a graph $G = (V, E)$, where each vertex is labeled by a function $f_v \in \mathcal{F}$, and $\pi : V \to \mathcal{F}$ is the labelling. The Holant problem on instance Ω is to evaluate $\mathrm{Holant}_\Omega = \sum_\sigma \prod_{v \in V} f_v(\sigma \mid_{E(v)})$, a sum over all edge assignments $\sigma : E \to \{0, 1\}$.

A function f_v can be represented by its truth table, which is a vector in $\mathbb{C}^{2^{\deg(v)}}$, or as a tensor in $(\mathbb{C}^2)^{\otimes \deg(v)}$. We also use f^α to denote the value $f(\alpha)$, where α is a binary string. A function $f \in \mathcal{F}$ is also called a *signature*. A symmetric signature f on k Boolean variables can be expressed as $[f_0, f_1, \ldots, f_k]$, where f_i is the value of f on inputs of Hamming weight i. In this paper, we consider symmetric signatures. Since a signature of arity k must be placed on a vertex of degree k, we can represent a signature of arity k by a labeled vertex with k ordered dangling edges. Throughout this paper, we do not distinguish these two views.

A Holant problem is parametrized by a set of signatures.

Definition 2.1. *Given a set of signatures $\mathcal{F}$, we define the counting problem* $\mathrm{Holant}(\mathcal{F})$ *as:*
Input: A signature grid $\Omega = (G, \mathcal{F}, \pi)$*;*
Output: Holant_Ω*.*

The following family Holant^* of Holant problems were investigated previously [11, 12]. This is the class of Holant problems in which all unary signatures are freely available.

Definition 2.2. *Given a set of signatures $\mathcal{F}$,* $\mathrm{Holant}^*(\mathcal{F})$ *denotes* $\mathrm{Holant}(\mathcal{F} \cup \mathcal{U})$*, where $\mathcal{U}$ is the set of all unary signatures.*

The family Holant^c of Holant problems (on Boolean variables) are defined analogously. The c stands for *constants* and refers to the signatures that can fix a variable to a constant of the domain.

Definition 2.3. *Given a set of signatures $\mathcal{F}$,* $\text{Holant}^c(\mathcal{F})$ *denotes* $\text{Holant}(\mathcal{F} \cup \{[0,1],[1,0]\})$.

A signature f of arity n is *degenerate* if there exist unary signatures $u_j \in \mathbb{C}^2$ $(1 \le j \le n)$ such that $f = u_1 \otimes \cdots \otimes u_n$. A symmetric degenerate signature has the from $u^{\otimes n}$. For such signatures, it is equivalent to replace it by n copies of the corresponding unary signatures. Replacing a signature $f \in \mathcal{F}$ by a constant multiple cf, where $c \neq 0$, does not change the complexity of $\text{Holant}(\mathcal{F})$. It introduces a global factor to Holant_Ω. Hence, for two signatures f, g of the same arity, we use $f \neq g$ to mean that these signatures are not equal in the projective space sense, i.e. not equal up to any nonzero constant multiple.

We say a signature set $\mathcal{F}$ is tractable (resp. #P-hard) if the corresponding counting problem $\text{Holant}(\mathcal{F})$ is tractable (resp. #P-hard). Similarly for a signature f, we say f is tractable (resp. #P-hard) if $\{f\}$ is. We follow the usual conventions about polynomial time Turing reduction $\le_T$ and polynomial time Turing equivalence $\equiv_T$.

2.2 Holographic Reduction

To introduce the idea of holographic reductions, it is convenient to consider bipartite graphs. For a general graph, we can always transform it into a bipartite graph while preserving the Holant value, as follows. For each edge in the graph, we replace it by a path of length 2. (This operation is called the *2-stretch* of the graph and yields the edge-vertex incident graph.) Each new vertex is assigned the binary EQUALITY signature $(=_2) = [1,0,1]$.

We use $\text{Holant}(\mathcal{R} \mid \mathcal{G})$ to denote the Holant problem on bipartite graphs $H = (U, V, E)$, where each signature for a vertex in U or V is from $\mathcal{R}$ or $\mathcal{G}$, respectively. An input instance for this bipartite Holant problem is a bipartite signature grid and is denoted by $\Omega = (H;\ \mathcal{R} \mid \mathcal{G};\ \pi)$. Signatures in $\mathcal{R}$ are considered as row vectors (or covariant tensors); signatures in $\mathcal{G}$ are considered as column vectors (or contravariant tensors) [20].

For a 2-by-2 matrix T and a signature set $\mathcal{F}$, define $T\mathcal{F} = \{g \mid \exists f \in \mathcal{F} \text{ of arity } n,\ g = T^{\otimes n} f\}$, similarly for $\mathcal{F}T$. Whenever we write $T^{\otimes n} f$ or $T\mathcal{F}$, we view the signatures as column vectors; similarly for $fT^{\otimes n}$ or $\mathcal{F}T$ as row vectors.

Let T be an invertible 2-by-2 matrix. The holographic transformation by T is the following operation: given a signature grid $\Omega = (H;\ \mathcal{R} \mid \mathcal{G};\ \pi)$, for the same graph H, we get a new grid $\Omega' = (H;\ \mathcal{R}T \mid T^{-1}\mathcal{G};\ \pi')$ by replacing each signature in $\mathcal{R}$ or $\mathcal{G}$ with the corresponding signature in $\mathcal{R}T$ or $T^{-1}\mathcal{G}$.

Theorem 2.4 (Valiant's Holant Theorem [44])**.** *If there is a holographic transformation mapping signature grid Ω to Ω', then* $\text{Holant}_\Omega = \text{Holant}_{\Omega'}$.

Therefore, an invertible holographic transformation does not change the complexity of the Holant problem in the bipartite setting. Furthermore, there is a special kind of holographic transformation, the orthogonal transformation, that preserves the binary equality and thus can be used freely in the standard setting.

Theorem 2.5 (Theorem 2.2 in [11])**.** *Suppose T is a 2-by-2 orthogonal matrix ($TT^{\mathtt{T}} = I_2$) and let $\Omega = (H, \mathcal{F}, \pi)$ be a signature grid. Under a holographic transformation by T, we get a new grid $\Omega' = (H, T\mathcal{F}, \pi)$ and* $\text{Holant}_\Omega = \text{Holant}_{\Omega'}$.

Since the complexity of signatures are equivalent up to a nonzero constant factor, we also call a transformation T such that $TT^{\mathtt{T}} = \lambda I$ for some $\lambda \neq 0$ an orthogonal transformation. Such transformations do not change the complexity of a problem.

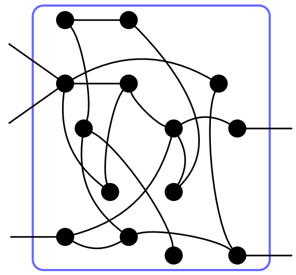


Figure 1: An $\mathcal{F}$-gate with 5 dangling edges.

2.3 Realization

One basic notion used throughout the paper is realization. We say a signature f is *realizable* or *constructable* from a signature set $\mathcal{F}$ if there is a gadget with some dangling edges such that each vertex is assigned a signature from $\mathcal{F}$, and the resulting graph, when viewed as a black-box signature with inputs on the dangling edges, is exactly f. If f is realizable from a set $\mathcal{F}$, then we can freely add f into $\mathcal{F}$ preserving the complexity.

Formally, such a notion is defined by an $\mathcal{F}$-gate [11, 12]. An $\mathcal{F}$-gate is similar to a signature grid $(H, \mathcal{F}, \pi)$ except that $H = (V, E, D)$ is a graph with some dangling edges D. The dangling edges define external variables for the $\mathcal{F}$-gate. (See Figure 1 for an example.) We denote the regular edges in E by $1, 2, \ldots, m$, and denote the dangling edges in D by $m + 1, \ldots, m + n$. Then we can define a function Γ for this $\mathcal{F}$-gate as

$$\Gamma(y_1, y_2, \ldots, y_n) = \sum_{x_1, x_2, \ldots, x_m \in \{0,1\}} H(x_1, x_2, \ldots, x_m, y_1, \ldots, y_n),$$

where $(y_1, y_2, \ldots, y_n) \in \{0, 1\}^n$ denotes an assignment on the dangling edges and $H(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n)$ denotes the value of the signature grid on an assignment of all edges, which is the product of evaluations at all internal vertices. We also call this function the signature Γ of the $\mathcal{F}$-gate. An $\mathcal{F}$-gate can be used in a signature grid as if it is just a single vertex with the particular signature.

Using the idea of $\mathcal{F}$-gates, we can reduce one Holant problem to another. Suppose g is the signature of some $\mathcal{F}$-gate. Then $\operatorname{Holant}(\mathcal{F} \cup \{g\}) \le_T \operatorname{Holant}(\mathcal{F})$. The reduction is quite simple. Given an instance of $\operatorname{Holant}(\mathcal{F} \cup \{g\})$, by replacing every appearance of g by the $\mathcal{F}$-gate, we get an instance of $\operatorname{Holant}(\mathcal{F})$. Since the signature of the $\mathcal{F}$-gate is g, the Holant values for these two signature grids are identical.

Although our main result is about symmetric signatures, some of our proofs utilize asymmetric signatures. When an asymmetric signature is used in a gadget, we place a diamond on the edge corresponding to the most significant index bit. The remaining index bits are in order of decreasing significance as one travels counterclockwise around the vertex. (See Figure 4 for an example.) Some of our gadget constructions are bipartite graphs. To highlight this structure, we use vertices of different shapes. Any time a gadget has a square vertex, it is assigned $[0, 1, 0]$. (See Figure 7 for an example.)

We note that even for a very simple signature set $\mathcal{F}$, the signatures for all $\mathcal{F}$-gates can be quite complicated and expressive.

2.4 #CSP and Its Tractable Signatures

An instance of #CSP($\mathcal{F}$) has the following bipartite view. We make a node for each variable and each constraint. Connect a variable node to a constraint node if the variable appears in the constraint function. This bipartite graph is also known as the *constraint graph*. Under this view, we can see that

$$\#\text{CSP}(\mathcal{F}) \equiv_T \text{Holant}\,(\mathcal{F} \mid \mathcal{EQ}) \equiv_T \text{Holant}(\mathcal{F} \cup \mathcal{EQ}),$$

where $\mathcal{EQ} = \{=_1, =_2, =_3, \dots\}$ is the set of equalities of all arities.

For a positive integer d, the problem #CSP^d($\mathcal{F}$) is similar to #CSP($\mathcal{F}$) except that every variable has to appear a multiple of d times. Thus we have

$$\#\text{CSP}^d(\mathcal{F}) \equiv_T \text{Holant}\,(\mathcal{F} \mid \mathcal{EQ}_d)\,,$$

where $\mathcal{EQ}_d = \{=_d, =_{2d}, =_{3d}, \dots\}$ is the set of equalities of arities that are a multiple of d.

For the #CSP framework, the following two sets of signatures are tractable [11].

Definition 2.6. *A k-ary function $f(x_1, \dots, x_k)$ is* affine *if it has the form*

$$\lambda \chi_{Ax=0} \cdot \sqrt{-1}^{\sum_{j=1}^{n} \langle \alpha_j, x \rangle},$$

where $\lambda \in \mathbb{C}$, $x = (x_1, x_2, \dots, x_k, 1)^T$, A is a matrix over $\mathbb{F}_2$, α_j is a vector over $\mathbb{F}_2$, and χ is a 0-1 indicator function such that $\chi_{Ax=0}$ is 1 iff $Ax = 0$. Note that the dot product $\langle \alpha_j, x \rangle$ is calculated over $\mathbb{F}_2$, while the summation $\sum_{j=1}^{n}$ on the exponent of $i = \sqrt{-1}$ is evaluated as a sum mod 4 of 0-1 terms. We use $\mathscr{A}$ to denote the set of all affine functions.

Definition 2.7. *A function is of* product type *if it can be expressed as a product of unary functions, binary equality functions* ($[1, 0, 1]$)*, and binary disequality functions* ($[0, 1, 0]$)*. We use $\mathscr{P}$ to denote the set of product type functions.*

An alternate definition for $\mathscr{P}$, implicit in [15], is the tensor closure of signatures with support on two entries of complement indices.

It is easy to see (cf. Lemma A.1 in the full version of [30]) that if f is a symmetric signature in $\mathscr{P}$, then f is either degenerate, binary disequality, or generalized equality (i.e. $[a, 0, \dots, 0, b]$ for $a, b \in \mathbb{C}$). Since our main dichotomy theorem is for symmetric signatures, we use $\mathscr{A}$ (resp. $\mathscr{P}$) to refer to the set of *symmetric* affine (resp. product-type) signatures. It is known that the set of nondegenerate symmetric signatures in $\mathscr{A}$ is contained in $\mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$, where $\mathscr{F}_1$, $\mathscr{F}_2$, and $\mathscr{F}_3$ are three families of signatures defined as

$$\mathscr{F}_1 = \left\{ \lambda \left([1, 0]^{\otimes k} + i^r [0, 1]^{\otimes k} \right) \mid \lambda \in \mathbb{C}, k = 1, 2, \dots, r = 0, 1, 2, 3 \right\},$$
$$\mathscr{F}_2 = \left\{ \lambda \left([1, 1]^{\otimes k} + i^r [1, -1]^{\otimes k} \right) \mid \lambda \in \mathbb{C}, k = 1, 2, \dots, r = 0, 1, 2, 3 \right\}, \text{ and}$$
$$\mathscr{F}_3 = \left\{ \lambda \left([1, i]^{\otimes k} + i^r [1, -i]^{\otimes k} \right) \mid \lambda \in \mathbb{C}, k = 1, 2, \dots, r = 0, 1, 2, 3 \right\}.$$

We explicitly list all the signatures in $\mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$ up to an arbitrary constant multiple from $\mathbb{C}$:

1. $[1,0,\ldots,0,\pm 1]$; $(\mathscr{F}_1, r=0,2)$
2. $[1,0,\ldots,0,\pm i]$; $(\mathscr{F}_1, r=1,3)$
3. $[1,0,1,0,\ldots,0 \text{ or } 1]$; $(\mathscr{F}_2, r=0)$
4. $[1,-i,1,-i,\ldots,(-i) \text{ or } 1]$; $(\mathscr{F}_2, r=1)$
5. $[0,1,0,1,\ldots,0 \text{ or } 1]$; $(\mathscr{F}_2, r=2)$
6. $[1,i,1,i,\ldots,i \text{ or } 1]$; $(\mathscr{F}_2, r=3)$
7. $[1,0,-1,0,1,0,-1,0,\ldots,0 \text{ or } 1 \text{ or } (-1)]$; $(\mathscr{F}_3, r=0)$
8. $[1,1,-1,-1,1,1,-1,-1,\ldots,1 \text{ or } (-1)]$; $(\mathscr{F}_3, r=1)$
9. $[0,1,0,-1,0,1,0,-1,\ldots,0 \text{ or } 1 \text{ or } (-1)]$; $(\mathscr{F}_3, r=2)$
10. $[1,-1,-1,1,1,-1,-1,1,\ldots,1 \text{ or } (-1)]$. $(\mathscr{F}_3, r=3)$

In the Holant framework, there are two corresponding signature sets that are tractable. A signature f (resp. a signature set $\mathcal{F}$) is $\mathscr{A}$-transformable if there exists a holographic transformation T such that $f \in T\mathscr{A}$ (resp. $\mathcal{F} \subseteq T\mathscr{A}$) and $[1,0,1]T^{\otimes 2} \in \mathscr{A}$. Similarly, a signature f (resp. a signature set $\mathcal{F}$) is $\mathscr{P}$-transformable if there exists a holographic transformation T such that $f \in T\mathscr{P}$ (resp. $\mathcal{F} \subseteq T\mathscr{P}$) and $[1,0,1]T^{\otimes 2} \in \mathscr{P}$. These two families are tractable because after a transformation by T, it is a tractable #CSP instance.

2.5 Some Known Dichotomies

Here we list several known dichotomies. Our main dichotomy theorem is a generalization of all of them. In order to clearly see this, we state the previous dichotomies using the language of this paper. In particular, some previous classifications are now presented differently using our new understanding.

The dichotomy for a single symmetric ternary signature is an important base case in the proof of our theorem.

Theorem 2.8 (Theorem 3 in [8]). *If $f = [f_0, f_1, f_2, f_3]$ is a non-degenerate, complex-valued signature, then* Holant(f) *is* #P*-hard unless f satisfies one of the following conditions, in which case the problem is in* P*:*

1. *f is $\mathscr{A}$- or $\mathscr{P}$-transformable;*
2. *For $\alpha \in \{2i, -2i\}$, $f_2 = \alpha f_1 + f_0$ and $f_3 = \alpha f_2 + f_1$.*

We also use the following theorem about edge-weighted signatures on k-regular graphs.

Theorem 2.9 (Theorem 3 in [10]). *Let $k \geq 3$ be an integer and suppose f is a non-degenerate, symmetric, complex-valued binary signature. Then* Holant$(f \mid =_k)$ *is* #P*-hard unless there exists a holographic transformation T such that $fT^{\otimes 2} = [1,0,1]$ and $\left((T^{-1})^{\otimes k}(=_k)\right)$ is $\mathscr{A}$- or $\mathscr{P}$-transformable, in which case the problem is in* P.

Theorem 2.9 is more conceptual, but the original statement, which is given in Theorem 2.9′, is more directly applicable.

Theorem 2.9′ (Theorem 3 in [10]). *Let $k \geq 3$ be an integer. Then* Holant$([f_0, f_1, f_2] \mid (=_k))$ *is* #P*-hard unless one of the following conditions hold, in which case the problem is in* P*:*

1. *$f_0 f_2 = f_1^2$;*

2. $f_0 = f_2 = 0$;
3. $f_1 = 0$;
4. $f_0 f_2 = -f_1^2$ and $f_0^{2k} = f_2^{2k}$.

The next theorem is a generalization of the Boolean #CSP dichotomy (where $d = 1$). Define $\mathcal{T}_k = \left\{ \left[\begin{smallmatrix} 1 & 0 \\ 0 & \omega \end{smallmatrix}\right] \mid \omega^k = 1 \right\}$.

Theorem 2.10 (Theorem IV.1 in [30])**.** *Let $d \geq 1$ be an integer and $\mathcal{F}$ be any set of symmetric, complex-valued signatures in Boolean variables. Then* $\#\mathrm{CSP}^d(\mathcal{F})$ *is* #P*-hard unless there exists $T \in \mathcal{T}_{4d}$ such that $T\mathcal{F} \subseteq \mathscr{P}$ or $T\mathcal{F} \subseteq \mathscr{A}$, in which case the problem is in* P.

The following three dichotomies are not directly used in this paper. We list them for comparison. First is the real-valued Holant dichotomy. Our results have no dependence on this dichotomy.

Theorem 2.11 (Theorem III.2 in [30])**.** *Let $\mathcal{F}$ be any set of symmetric, real-valued signatures in Boolean variables. Then* $\mathrm{Holant}(\mathcal{F})$ *is* #P*-hard unless $\mathcal{F}$ satisfies one of the following conditions, in which case the problem is in* P*:*

1. *Any non-degenerate signature in $\mathcal{F}$ is of arity at most 2;*
2. *$\mathcal{F}$ is $\mathscr{A}$- or $\mathscr{P}$-transformable.*

The other two dichotomies are the complex-valued Holant^* and Holant^c dichotomy theorems. Although we do not directly apply these, our results depend on them through Theorems 2.8, 2.9, and 2.10.

Theorem 2.12 (Theorem 3.1 in [11])**.** *Let $\mathcal{F}$ be any set of non-degenerate, symmetric, complex-valued signatures in Boolean variables. Then* $\mathrm{Holant}^*(\mathcal{F})$ *is* #P*-hard unless $\mathcal{F}$ satisfies one of the following conditions, in which case the problem is in* P*:*

1. *Any signature in $\mathcal{F}$ is of arity at most 2;*
2. *$\mathcal{F}$ is $\mathscr{P}$-transformable;*
3. *There exists $\alpha \in \{2i, -2i\}$, such that for any signature $f \in \mathcal{F}$ of arity n, for $0 \leq k \leq n-2$, we have $f_{k+2} = \alpha f_{k+1} + f_k$.*

Theorem 2.13 (Theorem 6 in [8])**.** *Let $\mathcal{F}$ be any set of symmetric, complex-valued signatures in Boolean variables. Then* $\mathrm{Holant}^c(\mathcal{F})$ *is* #P*-hard unless $\mathcal{F}$ satisfies one of the following conditions, in which case the problem is in* P*:*

1. *Any non-degenerate signature in $\mathcal{F}$ is of arity at most 2;*
2. *$\mathcal{F}$ is $\mathscr{P}$-transformable;*
3. *$\mathcal{F} \cup \{[1,0],[0,1]\}$ is $\mathscr{A}$-transformable;*
4. *There exists $\alpha \in \{2i, -2i\}$, such that for any non-degenerate signature $f \in \mathcal{F}$ of arity n, for $0 \leq k \leq n-2$, we have $f_{k+2} = \alpha f_{k+1} + f_k$.*

3 A Sampling of Problems

We illustrate the scope of our dichotomy theorem by several concrete problems. Some problems are naturally expressed with real weights, but they are linked inextricably to other problems that use complex weights. Sometimes the inherent link between two real-weighted problems is provided by a transformation through $\mathbb{C}$.

Problem: #VertexCover
Input: An undirected graph G.
Output: The number of vertex covers in G.

This classic problem is most naturally expressed as the real-weighted bipartite Holant problem $\text{Holant}\,([0,1,1] \mid \mathcal{EQ})$. A vertex assigned an equality signature forces all its incident edges to be assigned the same value; this is equivalent to these vertices being assigned a value themselves. The degree two vertices assigned the binary $\text{Or} = [0,1,1]$ should be thought of as an edge between its neighboring vertices. These edge-like vertices force at least one of its neighbors to be selected. The number of assignments satisfying these requirements is exactly the number of vertex covers.

To apply our dichotomy theorem, we perform a holographic transformation by $T = \left[\begin{smallmatrix} 0 & -i \\ 1 & i \end{smallmatrix}\right]$. To understand why we choose this particular T, let us express $[0,1,1]$ as

$$\begin{aligned} [0,1,1] = (0\ 1\ 1\ 1) = \left\{[1,1]^{\otimes 2} + [i,0]^{\otimes 2}\right\} &= \left\{[1,0]^{\otimes 2} + [0,1]^{\otimes 2}\right\} \begin{bmatrix} 1 & 1 \\ i & 0 \end{bmatrix}^{\otimes 2} \\ = (1\ 0\ 0\ 1)(T^{-1})^{\otimes 2} &= (=_2)(T^{-1})^{\otimes 2}. \end{aligned}$$

Thus, a holographic transformation by T yields

$$\begin{aligned} \text{Holant}\,([0,1,1] \mid \mathcal{EQ}) &\equiv_T \text{Holant}\left([0,1,1]T^{\otimes 2} \mid T^{-1}\mathcal{EQ}\right) \\ &\equiv_T \text{Holant}\left(=_2 \mid T^{-1}\mathcal{EQ}\right) \\ &\equiv_T \text{Holant}(T^{-1}\mathcal{EQ}). \end{aligned}$$

The equality signature of arity k in $\mathcal{EQ}$, a column vector denoted by $=_k$, is transformed by T^{-1} to

$$\begin{aligned} f_{(k)} = (T^{-1})^{\otimes k}(=_k) &= \begin{bmatrix} 1 & 1 \\ i & 0 \end{bmatrix}^{\otimes k} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\otimes k} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}^{\otimes k} \right\} \\ &= \begin{bmatrix} 1 \\ i \end{bmatrix}^{\otimes k} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\otimes k} = [2, i, -1, -i, 1, i, -1, -i, 1, i, \dots] \end{aligned}$$

of length $k+1$. By our main dichotomy, Theorem 5.1, $\text{Holant}(T^{-1}\mathcal{EQ})$ is #P-hard. Indeed, even $\text{Holant}(f_{(k)})$, the restriction of this problem to k-regular graphs is #P-hard for $k \ge 3$ by our single signature dichotomy, Theorem 9.1.

Problem: #λ-VertexCover
Input: An undirected graph G.
Output: $\displaystyle\sum_{C \in \mathcal{C}(G)} \lambda^{e(C)}$,

where $\mathcal{C}(G)$ denotes the set of all vertex covers of G, and $e(C)$ is the number of edges with both endpoints in the vertex cover C.

Our dichotomy also easily handles this edge-weighted vertex cover problem that is denoted by $\text{Holant}([0,1,\lambda] \mid \mathcal{EQ})$. Suppose $\lambda \neq 0$. On regular graphs, this problem is equivalent to the so-called *hardcore gas model*, which is the vertex-weighted problem denoted by $\text{Holant}([1,1,0] \mid \mathcal{F})$, where $\mathcal{F}$ consists of signatures of the form $[1,0,\ldots,0,\mu]$. By flipping 0 and 1, this is the same as $\text{Holant}([0,1,1] \mid \mathcal{F}')$ with $\mathcal{F}'$ containing $[\mu,0,\ldots,0,1]$. For k-regular graphs, we consider the diagonal transformation $T = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix}$, where $\lambda = 1/\mu^{1/k}$;

$$\begin{aligned}
\text{Holant}([0,1,\lambda] \mid =_k) &\equiv_T \text{Holant}\left([0,1,\lambda]T^{\otimes 2} \mid (T^{-1})^{\otimes k}(=_k)\right) \\
&\equiv_T \text{Holant}\left(\tfrac{1}{\lambda}[0,1,1] \mid [1,0,\cdots,0,\lambda^k]\right) \\
&\equiv_T \text{Holant}([0,1,1] \mid [\mu,0,\cdots,0,1]).
\end{aligned}$$

This problem, denoted by #k-λ-VertexCover, is also #P-hard for $k \geq 3$. To see this, apply the holographic transformation $T = \begin{bmatrix} 0 & -i\lambda \\ 1 & i \end{bmatrix}$ to the edge-weighted form of the problem. Then $[0,1,\lambda]$ is transformed into $\lambda \cdot (=_2)$ and $=_k$ is transformed to $g_{(\lambda,k)} = \frac{1}{\lambda^k}[\lambda^k+1, i, -1, -i, 1, \ldots]$. Since $\text{Holant}(g_{(\lambda,k)})$ is #P-hard by Theorem 9.1, we conclude that #k-λ-VertexCover is also #P-hard.

If $\lambda = 0$, then the above problem is $\text{Holant}([0,1,0] \mid \mathcal{EQ})$, which is tractable. However, the transformation T above is singular in this case. We can in fact apply another transformation $T' = \begin{bmatrix} 1-\frac{\lambda}{2} & -\left(1+\frac{\lambda}{2}\right)i \\ 1 & i \end{bmatrix}$ such that it transforms the problem $\text{Holant}([0,1,\lambda] \mid =_k)$ into $\text{Holant}(h_{(\lambda,k)})$ for some $h_{(\lambda,k)}$ regardless of whether $\lambda = 0$ or not. Then by applying Theorem 9.1, we reach the same conclusion that #λ-VertexCover is #P-hard on k-regular graphs when $\lambda \neq 0$. We note that when $\lambda = 0$, $T' = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} = \sqrt{2}Z^{-1}$, where $Z = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$ was used in Section 1.

We now consider some orientation problems.

Problem: #NoSinkOrientation
Input: An undirected graph G.
Output: The number of orientations of G such that each vertex has at least one outgoing edge.

This problem is denoted by $\text{Holant}([0,1,0] \mid \mathcal{F})$, where $\mathcal{F}$ consists of $f_{(k)} = [0,1,\ldots,1,1]$ for any arity k. Each degree two vertex on the left side of the bipartite graph must have its incident edges assigned different values. We associate an oriented edge between the neighbors of such vertices with the head on the side assigned 0 and the tail on the side assigned 1. This problem is #P-hard even over k-regular graphs provided $k \geq 3$. Just as with the bipartite form of the vertex cover problem, we do a holographic transformation to apply our dichotomy theorem. This time, we pick $T = \frac{1}{2}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} = \frac{1}{\sqrt{2}}Z^{-1}$, with $T^{-1} = \sqrt{2}Z = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$ and get

$$\begin{aligned}
\text{Holant}\left([0,1,0] \mid f_{(k)}\right) &\equiv_T \text{Holant}\left([0,1,0]T^{\otimes 2} \mid (T^{-1})^{\otimes k} f_{(k)}\right) \\
&\equiv_T \text{Holant}\left(\tfrac{1}{2}[1,0,1] \mid \widehat{f_{(k)}}\right) \\
&\equiv_T \text{Holant}(\widehat{f_{(k)}}),
\end{aligned}$$

where $\widehat{f_{(k)}} = [2^k-1, -i, 1, i, -1, \ldots]$. This is actually a special case (consider $-\widehat{f_{(k)}}$) of the #k-λ-VertexCover problem with $\lambda = 2e^{\pi i/k}$. Therefore, this problem is #P-hard. However, if we consider this problem modulo 2^k, $\widehat{f_{(k)}}$ becomes $[-1,-i,1,\dots]$, and belongs to one of the tractable cases in our dichotomy. Therefore, #NoSinkOrientation is tractable modulo 2^t, where t is the minimal degree of the input graph.

Problem: #NoSinkNoSourceOrientation

Input: An undirected graph G.

Output: The number of orientations of G such that each vertex has at least one incoming and one outgoing edge.

This problem is denoted by $\operatorname{Holant}([0,1,0] \mid \mathcal{F})$, where $\mathcal{F}$ consists of $f_{(k)} = [0,1,\dots,1,0]$ for any arity k. This problem is also #P-hard on k-regular graphs for $k \geq 3$. We pick the same T as in the previous problem and get

$$\begin{aligned}\operatorname{Holant}\left([0,1,0] \mid f_{(k)}\right) &\equiv_T \operatorname{Holant}\left([0,1,0]T^{\otimes 2} \mid (T^{-1})^{\otimes k} f_{(k)}\right)\\ &\equiv_T \operatorname{Holant}\left(\tfrac{1}{2}[1,0,1] \mid \widehat{f_{(k)}}\right)\\ &\equiv_T \operatorname{Holant}(\widehat{f_{(k)}}),\end{aligned}$$

where $\widehat{f_{(k)}} = [2^k-2,0,2,0,-2,\dots]$. Here we transform from one real-weighted Holant problem to another real-weighted Holant problem via a complex-weighted transformation. The hardness follows from Theorem 9.1. Like the previous problem, #NoSinkNoSourceOrientation is tractable modulo 2^t, where t is the minimal degree of the input graph.

Our dichotomy theorem also applies to a set of signatures, that is, different vertices may have different constraints.

Problem: #1In-Or-1Out-Orientation

Input: An undirected graph G.

Output: The number of orientations of G such that each vertex has exactly 1 incoming or exactly 1 outgoing edge.

This problem is denoted by $\operatorname{Holant}([0,1,0] \mid \mathcal{F})$, where the set $\mathcal{F}$ consists of signatures of the form $f = [0,1,0,\dots,0]$ and $g = [0,\dots,0,1,0]$. Once again, it is #P-hard on k-regular graphs for $k \geq 3$. We apply the same transformation as in the above two orientation problems. The resulting problem is $\operatorname{Holant}(\{\hat{f},\hat{g}\})$, where $\hat{f} = [k,(k-2)i,-(k-4),\dots]$ and $\hat{g} = [k,-(k-2)i,-(k-4),\dots]$ of arity k. In fact, the entries of $\hat{f}$ satisfy a second order recurrence relation with characteristic polynomial $(x-i)^2$ while the entries of $\hat{g}$ satisfy one with characteristic polynomial $(x+i)^2$. The hardness follows from Theorem 5.1. However, if we consider only one signature, either $\operatorname{Holant}(\hat{f})$ or $\operatorname{Holant}(\hat{g})$ is tractable. The problem $\operatorname{Holant}(\hat{f})$ is equivalent to the problem $\operatorname{Holant}([0,1,0] \mid [0,1,0,\dots,0])$, which is always 0 provided $k \geq 3$ by a simple counting argument. Similarly for $\operatorname{Holant}(\hat{g})$. Therefore, despite the complicated looking $\hat{f}$ and $\hat{g}$, the Holant value for any input graph using only $\hat{f}$ or $\hat{g}$ is always 0. These are what we call vanishing signatures. This is also an example where combining two vanishing signatures induces #P-hardness.

One sufficient condition for a signature to be vanishing is that its entries satisfy a second order recurrence relation with characteristic polynomial $(x \pm i)^2$. If the entries of a signature f satisfy a second order recurrence relation with characteristic polynomial $(x-a)^2$ for $a \neq \pm i$, then there exists an orthogonal holographic transformation such that f is transformed into a weighted matching signature.

Problem: #λ-WeightedMatching

Input: An undirected graph G.
Output: $\sum_{M\in\mathcal{M}(G)} \lambda^{v(M)}$,
where $\mathcal{M}(G)$ is the set of all matchings in G and $v(M)$ is the number of unmatched vertices in the matching M.

The Holant expression of this problem is $\operatorname{Holant}(\mathcal{F})$, where $\mathcal{F}$ consists of signatures of the form $[\lambda, 1, 0, \ldots, 0]$. When $\lambda = 0$, this problem counts perfect matchings, which is #P-hard even for bipartite graphs [42] but tractable over planar graphs [33]. When $\lambda = 1$, this problem counts general matchings. Vadhan [41] proved that counting general matchings is #P-hard over k-regular graphs for $k \geq 5$, but left open the question for $k = 4$. Theorem 9.1 shows that #λ-WeightedMatching is #P-hard, for any weight λ and on any k-regular graphs for $k \geq 3$. The power of our dichotomy theorem is such that it gives a sweeping classification for *all* such problems; the open case for $k = 4$ from [41] is merely a single *point* in the problem space.

4 Vanishing Signatures

Vanishing signatures were first introduced in [28] in the parity setting to denote signatures for which the Holant value is always zero modulo 2.

Definition 4.1. *A set of signatures $\mathcal{F}$ is called* vanishing *if the value* $\operatorname{Holant}_\Omega(\mathcal{F})$ *is zero for every signature grid Ω. A signature f is called* vanishing *if the singleton set $\{f\}$ is vanishing.*

In this section, we characterize all sets of symmetric vanishing signatures. First we observe that a simple lemma (Lemma 6.2 in [28]) from the parity setting works over any field $\mathbb{F}$, with the same proof. It also works for general, not necessarily symmetric, signatures. Let $f + g$ denote the entry-wise addition of two signatures f and g with the same arity, i.e. $(f + g)_\ell = f_\ell + g_\ell$ for any index ℓ.

Lemma 4.2. *Let $\mathcal{F}$ be a vanishing signature set. If a signature f can be realized by a gadget using signatures in $\mathcal{F}$, then $\mathcal{F} \cup \{f\}$ is also vanishing. If f and g are two signatures in $\mathcal{F}$ of the same arity, then $\mathcal{F} \cup \{f + g\}$ is vanishing as well.*

Obviously, the identically zero signature, in which all entries are 0, is vanishing. This is trivial. However, we show that the concept of vanishing signatures is not trivial. Notice that the unary signature $[1, i]$ when connected to another $[1, i]$ has a Holant value 0. Consider a signature set $\mathcal{F}$ where every signature of arity n is degenerate. That is, every signature of arity n is a tensor product of unary signatures. Moreover, for each signature, suppose that more than half of the unary signatures in the tensor product are $[1, i]$. For any signature grid Ω with signatures from $\mathcal{F}$, it can be decomposed into many pairs of unary signatures. The total Holant value is the product of the Holant on each pair. Since more than half of the unaries in each signature are $[1, i]$, more than half of the unaries in Ω are $[1, i]$. Then two $[1, i]$'s must be paired up and hence $\operatorname{Holant}_\Omega = 0$. Thus, all such signatures form a vanishing set. We also observe that this argument holds when $[1, i]$ is replaced by $[1, -i]$.

These signatures described above are generally not symmetric and our present aim is to characterize symmetric vanishing signatures. To this end, we define the following symmetrization operation.

Definition 4.3. *Let S_n be the symmetric group of degree n. Then for positive integers t and n with $t \leq n$ and unary signatures $v, v_1, \ldots, v_{n-t}$, we define*

$$\operatorname{Sym}_n^t(v; v_1, \ldots, v_{n-t}) = \sum_{\pi \in S_n} \bigotimes_{k=1}^{n} u_{\pi(k)},$$

where the ordered sequence $(u_1, u_2, \ldots, u_n) = (\underbrace{v, \ldots, v}_{t \text{ copies}}, v_1, \ldots, v_{n-t})$.

Note that we include redundant permutations of v in the definition. Equivalent v_i's also induce redundant permutations. These redundant permutations simply introduce a nonzero constant factor, which does not change the complexity. However, the allowance of redundant permutations simplifies our calculations. An illustrative example of Definition 4.3 is

$$\begin{aligned}\operatorname{Sym}_3^2([1,i];[a,b]) &= 2[a,b] \otimes [1,i] \otimes [1,i] + 2[1,i] \otimes [a,b] \otimes [1,i] + 2[1,i] \otimes [1,i] \otimes [a,b] \\ &= 2[3a, 2ia+b, -a+2ib, -3b].\end{aligned}$$

Definition 4.4. *A nonzero symmetric signature f of arity n has* positive vanishing degree *$k \geq 1$, which is denoted by $\operatorname{vd}^+(f) = k$, if $k \leq n$ is the largest positive integer such that there exists $n-k$ unary signatures $v_1, \ldots, v_{n-k}$ satisfying*

$$f = \operatorname{Sym}_n^k([1,i]; v_1, \ldots, v_{n-k}).$$

If f cannot be expressed as such a symmetrization form, we define $\operatorname{vd}^+(f) = 0$. If f is the all zero signature, define $\operatorname{vd}^+(f) = n+1$.

We define negative vanishing degree vd^- *similarly, using $-i$ instead of i.*

Notice that it is possible for a signature f to have both $\operatorname{vd}^+(f)$ and $\operatorname{vd}^-(f)$ nonzero. For example, $f = [1,0,1]$ has $\operatorname{vd}^+(f) = \operatorname{vd}^-(f) = 1$.

By the discussion above and Lemma 4.2, we know that for a signature f of arity n, if $\operatorname{vd}^\sigma(f) > \frac{n}{2}$ for some $\sigma \in \{+,-\}$, then f is a vanishing signature. This argument is easily generalized to a set of signatures.

Definition 4.5. *For $\sigma \in \{+,-\}$, we define $\mathscr{V}^\sigma = \{f \mid 2\operatorname{vd}^\sigma(f) > \operatorname{arity}(f)\}$.*

Lemma 4.6. *For a set of symmetric signatures $\mathcal{F}$, if $\mathcal{F} \subseteq \mathscr{V}^+$ or $\mathcal{F} \subseteq \mathscr{V}^-$, then $\mathcal{F}$ is vanishing.*

In Theorem 4.13, we show that these two sets capture all symmetric vanishing signature sets.

4.1 Characterizing Vanishing Signatures using Recurrence Relations

Now we give an equivalent characterization of vanishing signatures.

Definition 4.7. *An arity n symmetric signature of the form $f = [f_0, f_1, \ldots, f_n]$ is in $\mathscr{R}_t^+$ for a nonnegative integer $t \geq 0$ if $t > n$ or for any $0 \leq k \leq n-t$, $f_k, \ldots, f_{k+t}$ satisfy the recurrence relation*

$$\binom{t}{t} i^t f_{k+t} + \binom{t}{t-1} i^{t-1} f_{k+t-1} + \cdots + \binom{t}{0} i^0 f_k = 0. \tag{2}$$

We define $\mathscr{R}_t^-$ similarly but with $-i$ in place of i in (2).

It is easy to see that $\mathscr{R}_0^+ = \mathscr{R}_0^-$ is the set of all zero signatures. Also, for $\sigma \in \{+,-\}$, we have $\mathscr{R}_t^\sigma \subseteq \mathscr{R}_{t'}^\sigma$ when $t \le t'$. By definition, if $\operatorname{arity}(f) = n$ then $f \in \mathscr{R}_{n+1}^\sigma$.

Let $f = [f_0, f_1, \ldots, f_n] \in \mathscr{R}_t^+$ with $0 < t \le n$. Then the characteristic polynomial of its recurrence relation is $(1+xi)^t$. Thus there exists a polynomial $p(x)$ of degree at most $t-1$ such that $f_k = i^k p(k)$, for $0 \le k \le n$. This statement extends to $\mathscr{R}_{n+1}^+$ since a polynomial of degree n can interpolate any set of $n+1$ values. Furthermore, such an expression is unique. If there are two polynomials $p(x)$ and $q(x)$, both of degree at most n, such that $f_k = i^k p(k) = i^k q(k)$ for $0 \le k \le n$, then $p(x)$ and $q(x)$ must be the same polynomial. Now suppose $f_k = i^k p(k)$ $(0 \le k \le n)$ for some polynomial p of degree at most $t-1$, where $0 < t \le n$. Then f satisfies the recurrence (2) of order t. Hence $f \in \mathscr{R}_t^+$.

Thus $f \in \mathscr{R}_{t+1}^+$ iff there exists a polynomials $p(x)$ of degree at most t such that $f_k = i^k p(k)$ $(0 \le k \le n)$, for all $0 \le t \le n$. For $\mathscr{R}_{t+1}^-$, just replace i by $-i$.

Definition 4.8. *For a nonzero symmetric signature f of arity n, it is of* positive *(resp.* negative*)* recurrence degree *$t \le n$, denoted by $\operatorname{rd}^+(f) = t$ (resp. $\operatorname{rd}^-(f) = t$), if and only if $f \in \mathscr{R}_{t+1}^+ - \mathscr{R}_t^+$ (resp. $f \in \mathscr{R}_{t+1}^- - \mathscr{R}_t^-$). If f is the all zero signature, we define $\operatorname{rd}^+(f) = \operatorname{rd}^-(f) = -1$.*

Note that although we call it the recurrence degree, it refers to a special kind of recurrence relation. For any nonzero symmetric signature f, by the uniqueness of the representing polynomial $p(x)$, it follows that $\operatorname{rd}^\sigma(f) = t$ iff $\deg(p) = t$, where $0 \le t \le n$. We remark that $\operatorname{rd}^\sigma(f)$ is the minimum integer t such that f does *not* belong to $\mathscr{R}_t^\sigma$. Also, for an arity n signature f, $\operatorname{rd}^\sigma(f) = n$ if and only if f does not satisfy any such recurrence relation (2) of order $t \le n$ for $\sigma \in \{+,-\}$.

Lemma 4.9. *Let $f = [f_0, \ldots, f_n]$ be a symmetric signature of arity n, not identically 0. Then for any nonnegative integer $0 \le t < n$ and $\sigma \in \{+,-\}$, the following are equivalent:*

(i) There exist t unary signatures $v_1, \ldots, v_t$, such that

$$f = \operatorname{Sym}_n^{n-t}([1, \sigma i]; v_1, \ldots, v_t). \tag{3}$$

(ii) $f \in \mathscr{R}_{t+1}^\sigma$.

Proof. We consider $\sigma = +$ since the other case is similar, so let $v = [1, i]$.

We start with $(i) \implies (ii)$ and proceed via induction on both t and n. For the first base case of $t = 0$, $\operatorname{Sym}_n^n(v) = [1, i]^{\otimes n} = [1, i, -1, -i, \ldots, i^n]$, so $f_{k+1} = i f_k$ for all $0 \le k \le n-1$ and $f \in \mathscr{R}_1^+$.

The other base case is that $t = n-1$. Let $\operatorname{Sym}_n^1(v; v_1, \ldots, v_t) = [f_0, \ldots, f_n]$ where $v_i = [a_i, b_i]$ for $1 \le i \le t$, and $S = i^n f_n + \cdots + \binom{n}{1} i f_1 + \binom{n}{0} i^0 f_0$. We need to show that $S = 0$. First notice that any entry in f is a linear combination of terms of the form $a_{i_1} a_{i_2} \cdots a_{i_{n-1-k}} b_{j_1} \cdots b_{j_k}$, where $0 \le k \le n-1$, and $\{i_1, \ldots, i_{n-1-k}, j_1, \ldots, j_k\} = \{1, 2, \ldots, n-1\}$. Thus S is a linear combination of such terms as well. Now we compute the coefficient of each of these terms in S.

Each term $a_{i_1} a_{i_2} \cdots a_{i_{n-1-k}} b_{j_1} \cdots b_{j_k}$ appears twice in S, once in f_k and the other time in f_{k+1}. In f_k, the coefficient is $k!(n-k)!$, and in f_{k+1}, it is $i(k+1)!(n-k-1)!$. Thus, its coefficient in S is

$$\binom{n}{k+1} i^{k+1} i (k+1)!(n-k-1)! + \binom{n}{k} i^k k!(n-k)! = 0.$$

The above computation works for any such term due to the symmetry of f, so all coefficients in S are 0, which means that $S = 0$.

Now assume for any $t' < t$ or for the same t and any $n' < n$, the statement holds. For (n, t), where $n > t+1$, assume that $f = [f_0, \ldots, f_n] = \mathrm{Sym}_n^{n-t}(v; v_1, \ldots, v_t)$, $g = \mathrm{Sym}_{n-1}^{n-t-1}(v; v_1, \ldots, v_t) = [g_0, \ldots, g_{n-1}]$, and for any $1 \le j \le t$, $h^{(j)} = \mathrm{Sym}_{n-1}^{n-t}(v; v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_t) = [h_0^{(j)}, \ldots, h_{n-1}^{(j)}]$. By the induction hypothesis, g satisfies the recurrence relation of order $t+1$, namely $g \in \mathscr{R}_{t+1}^+$. Also for any j, $h^{(j)}$ satisfies the recurrence relation of order t, namely $h^{(j)} \in \mathscr{R}_t^+ \subseteq \mathscr{R}_{t+1}^+$.

We have the recurrence relation

$$\begin{aligned}\mathrm{Sym}_n^{n-t}(v; v_1, \ldots, v_t) =& (n-t)v \otimes \mathrm{Sym}_{n-1}^{n-t-1}(v; v_1, \ldots, v_t) \\ &+ \sum_{j=1}^{t} v_j \otimes \mathrm{Sym}_{n-1}^{n-t}(v; v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_t).\end{aligned} \tag{4}$$

By equation (4), the entry of weight k in f for any $k > 0$ is

$$f_k = (n-t)ig_{k-1} + \sum_{j=1}^{t} b_j h_{k-1}^{(j)}.$$

We know that $\{g_i\}$ and $\{h_i^{(j)}\}$ satisfy the recurrence relation (2) of order $t+1$. Thus, their linear combination $\{f_i\}$ also satisfies the recurrence relation (2) starting from $i = k > 0$.

We also observe that by equation (4), the entry of weight k in f for any $k < n$ is

$$f_k = (n-t)g_k + \sum_{j=1}^{t} a_j h_k^{(j)}.$$

Since $t < n-1$, by the same argument again, the recurrence relation (2) holds for f when $k = 0$ as well.

Now we show $(ii) \implies (i)$. Notice that we only need to find unary signatures $\{v_i\}$ for $1 \le i \le t$ such that $\mathrm{Sym}_n^{n-t}(v; v_1, \ldots, v_t)$ matches the first $t+1$ entries of f. The theorem follows from this since we have shown that $\mathrm{Sym}_n^{n-t}(v; v_1, \ldots, v_t)$ satisfies the recurrence relation of order $t+1$ and any such signature is determined by the first $t+1$ entries.

We show that there exist $v_i = [a_i, b_i]$ $(1 \le i \le t)$ satisfying the above requirement. Since f is not identically 0, by equation (2), some nonzero term occurs among $\{f_0, \ldots, f_t\}$. Let $f_s \neq 0$, for $0 \le s \le t$, be the first nonzero term. By a nonzero constant multiplier, we may normalize $f_s = s!(n-s)!$, and set $v_j = [0, 1]$, for $1 \le j \le s$ (which is vacuous if $s = 0$), and set $v_{s+j} = [1, b_{s+j}]$, for $1 \le j \le t-s$ (which is vacuous if $s = t$). Let F be the function defined in equation (3). Then $F_k = f_k = 0$ for $0 \le k < s$ (which is vacuous if $s = 0$). By expanding the symmetrization function, for $s \le k \le t$, we get

$$F_k = k!(n-k)! \sum_{j=0}^{k-s} \binom{n-t}{k-s-j} \Delta_j i^{k-s-j},$$

where Δ_j is the elementary symmetric polynomial in $\{b_{s+1}, \ldots, b_t\}$ of degree j for $0 \le j \le t-s$. By definition, $\Delta_0 = 1$ and $F_s = f_s$. Setting $F_k = f_k$ for $s+1 \le k \le t$, this is a linear equation system on Δ_j $(1 \le j \le t-s)$, with a triangular matrix and nonzero diagonals. From this, we know that all Δ_j's are uniquely determined by $\{f_{s+1}, \ldots, f_t\}$. Moreover, $\{b_{s+1}, \ldots, b_t\}$ are the roots of the equation $\sum_{j=0}^{t-s} (-1)^j \Delta_j x^{t-j} = 0$. Thus $\{b_{s+1}, \ldots, b_t\}$ are also uniquely determined by $\{f_{s+1}, \ldots, f_t\}$ up to a permutation. $\square$

Corollary 4.10. *If f is a symmetric signature and $\sigma \in \{+,-\}$, then* $\operatorname{vd}^\sigma(f) + \operatorname{rd}^\sigma(f) = \operatorname{arity}(f)$.

Thus we have an equivalent form of $\mathscr{V}^\sigma$ for $\sigma \in \{+,-\}$. Namely,

$$\mathscr{V}^\sigma = \{f \mid 2\operatorname{rd}^\sigma(f) < \operatorname{arity}(f)\}.$$

4.2 Characterizing Vanishing Signature Sets

Now we show that $\mathscr{V}^+$ and $\mathscr{V}^-$ capture all symmetric vanishing signature sets. To begin, we show that a vanishing signature set cannot contain both types of nontrivial vanishing signatures.

Lemma 4.11. *Let $f_+ \in \mathscr{V}^+$ and $f_- \in \mathscr{V}^-$. If neither f_+ nor f_- is the zero signature, then the signature set $\{f_+, f_-\}$ is not vanishing.*

Proof. Let $\operatorname{arity}(f_+) = n$ and $\operatorname{rd}^+(f_+) = t$, so $2t < n$. Consider the gadget with two vertices and $2t$ edges between two copies of f_+. (See Figure 2 for an example of this gadget.) View f_+ in the symmetrized form. Since $\operatorname{vd}^+(f_+) = n - t$, in each term, there are $n - t$ many $[1, i]$'s and t many unary signatures not equal to (a multiple of) $[1, i]$. This is a superposition of many degenerate signatures. Then the only non-vanishing contributions come from the cases where the $n - 2t$ dangling edges on both sides are all assigned $[1, i]$, while inside, the t copies of $[1, i]$ pair up with t unary signatures not equal to $[1, i]$ from the other side perfectly. Notice that for any such contribution, the Holant value of the inside part is always the same constant and this constant is not zero because $[1, i]$ paired up with any unary signature other than (a multiple of) $[1, i]$ is not zero. Then the superposition of all of the permutations is a degenerate signature $[1, i]^{\otimes 2(n-2t)}$ up to a nonzero constant factor.

Similarly, we can do this for f_- of arity n' and $\operatorname{rd}^-(f_-) = t'$, where $2t' < n'$, and get a degenerate signature $[1, -i]^{\otimes 2(n'-2t')}$, up to a nonzero constant factor. Then form a bipartite signature grid with $(n' - 2t')$ vertices on one side, each assigned $[1, i]^{\otimes 2(n-2t)}$, and $(n - 2t)$ vertices on the other side, each assigned $[1, -i]^{\otimes 2(n'-2t')}$. Connect edges between the two sides arbitrarily as long as it is a 1-1 correspondence. The resulting Holant is a power of 2, which is not vanishing. □

Lemma 4.12. *Every symmetric vanishing signature is in $\mathscr{V}^+ \cup \mathscr{V}^-$.*

Proof. Let f be a symmetric vanishing signature. We prove this by induction on n, the arity of f. For $n = 1$, by connecting $f = [f_0, f_1]$ to itself, we have $f_0^2 + f_1^2 = 0$. Then up to a constant factor, we have either $f = [1, i]$ or $f = [1, -i]$. The lemma holds.

For $n = 2$, first we do a self loop. The Holant is $f_0 + f_2$. Also, we can connect two copies of f, in which case the Holant is $f_0^2 + 2f_1^2 + f_2^2$. Since f is vanishing, both $f_0 + f_2 = 0$ and $f_0^2 + 2f_1^2 + f_2^2 = 0$. Solving them, we get $f = [1, i, -1] = [1, i]^{\otimes 2}$ or $[1, -i, -1] = [1, -i]^{\otimes 2}$ up to a constant factor.

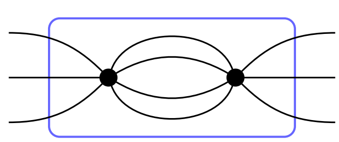


Figure 2: Example of a gadget used to create a degenerate vanishing signature from some general vanishing signature. This example is for a signature of arity 7 and recurrence degree 2, which is assigned to both vertices.

Now assume $n > 2$ and the lemma holds for any signature of arity $k < n$. Let $f = [f_0, f_1, \ldots, f_n]$ be a vanishing signature. A self loop on f gives $f' = [f'_0, f'_1, \ldots, f'_{n-2}]$, where $f'_j = f_j + f_{j+2}$ for $0 \le j \le n-2$. Since f is vanishing, f' is vanishing as well. By the induction hypothesis, $f' \in \mathscr{V}^+ \cup \mathscr{V}^-$.

If f' is a zero signature, then we have $f_j + f_{j+2} = 0$ for $0 \le j \le n-2$. This means that the f_j's satisfy a recurrence relation with characteristic polynomial x^2+1, so we have $f_j = ai^j + b(-i)^j$ for some a and b. Then we perform a holographic transformation with $Z = \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]$,

$$\begin{aligned} \operatorname{Holant}(=_2 \mid f) &\equiv_T \operatorname{Holant}\left([1,0,1]Z^{\otimes 2} \mid (Z^{-1})^{\otimes n} f\right) \\ &\equiv_T \operatorname{Holant}\left(2[0,1,0] \mid \hat{f}\right), \end{aligned}$$

where $\hat{f} = [a, 0, \ldots, 0, b]$. The problem $\operatorname{Holant}\left(2[0,1,0] \mid \hat{f}\right)$ is a weighted version of testing if a graph is bipartite. Now consider a graph with only two vertices, both assigned f, and n edges between them. The Holant of this graph is $2ab$. However, we know that it must be vanishing, so $ab = 0$. If $a = 0$, then $f \in \mathscr{V}^+$. Otherwise, $b = 0$ and $f \in \mathscr{V}^-$.

Now suppose that f' is in $\mathscr{V}^+ \cup \mathscr{V}^-$ but is not a zero signature. We consider $f' \in \mathscr{V}^+$ since the other case is similar. Then $\operatorname{rd}^+(f') = t$, so $2t < n-2$. Consider the gadget which has only two vertices, both assigned f', and has $2t$ edges between them. (See Figure 2 for an example of this gadget.) It forms a signature of degree $d = 2(n-2-2t)$. This gadget is valid because $n-2 > 2t$. By the combinatorial view as in the proof of Lemma 4.11, this signature is $[1,i]^{\otimes d}$.

Moreover, $\operatorname{rd}^+(f') = t$ implies that the entries of f' satisfy a recurrence of order $t+1$. Replacing f'_j by $f_j + f_{j+2}$, we get a recurrence relation for the entries of f with characteristic polynomial $(x^2+1)(x-i)^t = (x+i)(x-i)^{t+1}$. Thus, $f_j = i^j p(j) + c(-i)^j$ for some polynomial $p(x)$ of degree at most $t+1$ and some constant c. It suffices to show that $c = 0$ since $2(t+1) < n$ as $2t < n-2$.

Consider the signature $h = [h_0, \ldots, h_{n-1}]$ created by connecting f with a single unary signature $[1, i]$. For any $(n-1)$-regular graph $G = (V, E)$ with h assigned to every vertex, we can define a duplicate graph of $(d+1)|V|$ vertices as follows. First for each $v \in V$, define vertices v', $v_1, \ldots, v_d$. For each i, $1 \le i \le d$, we make a copy of G on $\{v_i \mid v \in V\}$, i.e., for each edge $(u, v) \in E$, include the edge (u_i, v_i) in the new graph. Next for each $v \in V$, we introduce edges between v' and v_i for all $1 \le i \le d$. For each $v \in V$, assign the degenerate signature $[1,i]^{\otimes d}$ that we just constructed to the vertices v'; assign f to all the vertices $v_1, \ldots, v_d$. Assume the Holant of the original graph G with h assigned to every vertex is H. Then for the new graph with the given signature assignments, the Holant is H^d. By our assumption, f is vanishing, so $H^d = 0$. Thus, $H = 0$. This holds for any graph G, so h is vanishing.

Notice that $h_k = f_k + i f_{k+1}$ for any $0 \le k \le n-1$. If h is identically zero, then $f_k + i f_{k+1} = 0$ for any $0 \le k \le n-1$, which means $f = [1, i]^{\otimes n}$ up to a constant factor and we are done. Suppose h is not zero. By the inductive hypothesis, $h \in \mathscr{V}^+ \cup \mathscr{V}^-$. We claim h cannot be from $\mathscr{V}^-$. This is because, although we do not directly construct h from f, we can always realize it by the method depicted in the previous paragraph. Therefore the set $\{f', h\}$ is vanishing. As both f' and h are nonzero, and $f' \in \mathscr{V}^+$, we have $h \notin \mathscr{V}^-$, by Lemma 4.11.

Hence h is in $\mathscr{V}^+$. Then there exists a polynomial $q(x)$ of degree at most $t' = \left\lfloor \frac{n-1}{2} \right\rfloor$ such that $h_k = i^k q(k)$, for any $0 \le k \le n-1$. Since $2t < n-2$, we have $t \le t'$. On the other hand,

$h_k = f_k + i f_{k+1}$ for any $0 \le k \le n-1$, so we have

$$\begin{aligned} h_k &= f_k + i f_{k+1} \\ &= i^k p(k) + c(-i)^k + i\left(i^{k+1} p(k+1) + c(-i)^{k+1}\right) \\ &= i^k \left(p(k) - p(k+1)\right) + 2c(-i)^k \\ &= i^k r(k) + 2c(-i)^k \\ &= i^k q(k), \end{aligned}$$

where $r(x) = p(x) - p(x+1)$ is another polynomial of degree at most t. Then we have

$$q(k) - r(k) = 2c(-1)^k,$$

which holds for all $0 \le k \le n-1$. Notice that the left hand side is a polynomial of degree at most t', call it $s(x)$. However, for all even $k \in \{0, \ldots, n-1\}$, $s(k) = 2c$. There are exactly $\lceil \frac{n}{2} \rceil > \lfloor \frac{n-1}{2} \rfloor = t'$ many even k within the range $\{0, \ldots, n-1\}$. Thus $s(x) = 2c$ for any x. Now we pick $k = 1$, so $s(1) = -2c = 2c$, which implies $c = 0$. This completes the proof. □

Combining Lemma 4.6, Lemma 4.11, and Lemma 4.12, we obtain the following theorem that characterizes all symmetric vanishing signature sets.

Theorem 4.13. *Let $\mathcal{F}$ be a set of symmetric signatures. Then $\mathcal{F}$ is vanishing if and only if $\mathcal{F} \subseteq \mathscr{V}^+$ or $\mathcal{F} \subseteq \mathscr{V}^-$.*

We note that some particular categories of tractable cases in previous dichotomies (case 2 of Theorem 2.8, case 3 of Theorem 2.12, and case 4 of Theorem 2.13) are in $\mathscr{R}_2^{\pm}$.

To finish this section, we prove some useful properties regarding vanishing and recurrence degrees in the construction of signatures. For two symmetric signatures f and g such that $\operatorname{arity}(f) \ge \operatorname{arity}(g)$, let $\langle f, g \rangle = \langle g, f \rangle$ denote the signature that results after connecting all edges of g to f. (If $\operatorname{arity}(f) = \operatorname{arity}(g)$, then $\langle f, g \rangle$ is a constant, which can be viewed as a signature of arity 0.)

Lemma 4.14. *For $\sigma \in \{+, -\}$, suppose symmetric signatures f and g satisfy $\operatorname{vd}^\sigma(g) = 0$ and $\operatorname{arity}(f) - \operatorname{arity}(g) \ge \operatorname{rd}^\sigma(f)$. Then $\operatorname{rd}^\sigma(\langle f, g \rangle) = \operatorname{rd}^\sigma(f)$.*

Proof. We consider $\sigma = +$ since the case $\sigma = -$ is similar. Let $\operatorname{arity}(f) = n$, $\operatorname{arity}(g) = m$, and $\operatorname{rd}^+(f) = t$. Denote the signature $\langle f, g \rangle$ by f'.

If $t = -1$, then f is identically 0 and so is f'. Hence $\operatorname{rd}^+(f') = -1$.

Suppose $t \ge 0$. Then we have $f_k = i^k p(k)$ where $p(x)$ is a polynomial of degree exactly t. Also $\operatorname{arity}(f') = n - m \ge t$. We have

$$\begin{aligned} f'_k &= \sum_{j=0}^{m} \binom{m}{j} f_{k+j} g_j \\ &= i^k \sum_{j=0}^{m} \binom{m}{j} p(k+j) i^j g_j \\ &= i^k q(k), \end{aligned}$$

where $q(k) = \sum_{j=0}^{m} \binom{m}{j} p(k+j) i^j g_j$ is a polynomial in k. Notice that $\operatorname{vd}^+(g) = 0$. Then $\operatorname{rd}^+(g) = m$ and $g \notin \mathscr{R}_m^+$. Thus $\sum_{j=0}^{m} \binom{m}{j} i^j g_j \ne 0$. Then the leading coefficient of degree t in the polynomial $q(k)$ is not zero. However, $\operatorname{arity}(f') \ge t$. Thus $\operatorname{rd}^+(f') = t$ as well. □

Lemma 4.15. *For $\sigma \in \{+, -\}$, let f be a nonzero symmetric signature and suppose that f' is obtained from f by a self loop. If* $\mathrm{vd}^\sigma(f) > 0$, *then* $\mathrm{vd}^\sigma(f) - \mathrm{vd}^\sigma(f') = \mathrm{rd}^\sigma(f) - \mathrm{rd}^\sigma(f') = 1$.

Proof. We may assume $\sigma = +$, $\mathrm{arity}(f) = n$, and $\mathrm{rd}^+(f) = t$. Since f is not the zero signature, $t \geq 0$. Also since $\mathrm{vd}^+(f) > 0$, $t = n - \mathrm{vd}^+(f) < n$. By assumption, we have $f_k = i^k p(k)$, where $p(x)$ is a polynomial of degree exactly t. Then we have

$$\begin{aligned} f'_k &= f_k + f_{k+2} \\ &= i^k(p(k) - p(k+2)) \\ &= i^k q(k), \end{aligned}$$

where $q(k) = p(k) - p(k+2)$ is a polynomial in k. If $t = 0$, then $p(x)$ is a constant polynomial and $q(x)$ is identically zero. Then $\mathrm{rd}^+(f') = -1$ by definition and $\mathrm{rd}^+(f) - \mathrm{rd}^+(f') = 1$ holds. Suppose $t > 0$, then in $q(k)$, the term of degree t has a zero coefficient, but the term of degree $t-1$ is nonzero. So $q(x)$ has degree exactly $t - 1 \leq n - 2 = \mathrm{arity}(f')$. Thus $\mathrm{rd}^+(f') = t - 1$. Notice that $\mathrm{arity}(f) - \mathrm{arity}(f') = 2$, then $\mathrm{vd}^+(f) - \mathrm{vd}^+(f') = 1$ as well. □

Moreover, the set of vanishing signatures is closed under orthogonal transformations. This is because under any orthogonal transformation, the unary signatures $[1, i]$ and $[1, -i]$ are either invariant or transformed into each other. Then considering the symmetrized form of any signature, we have the following lemma.

Lemma 4.16. *For a symmetric signature f of arity n, $\sigma \in \{+, -\}$, and an orthogonal matrix $T \in \mathbb{C}^{2\times 2}$, either* $\mathrm{vd}^\sigma(f) = \mathrm{vd}^\sigma(T^{\otimes n} f)$ *or* $\mathrm{vd}^\sigma(f) = \mathrm{vd}^{-\sigma}(T^{\otimes n} f)$.

4.3 Characterizing Vanishing Signatures via Holographic Transformation

There is another explanation for the vanishing signatures. Given an $f \in \mathscr{V}^+$ with $\mathrm{arity}(f) = n$ and $\mathrm{rd}^+(f) = d$, we perform a holographic transformation with $Z = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]$,

$$\begin{aligned} \mathrm{Holant}\,(=_2 \mid f) &\equiv_T \mathrm{Holant}\left([1,0,1]Z^{\otimes 2} \mid (Z^{-1})^{\otimes n} f\right) \\ &\equiv_T \mathrm{Holant}\left([0,1,0] \mid \hat{f}\right), \end{aligned}$$

where $\hat{f}$ is of the form $[\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_d, 0, \ldots, 0]$, and $\hat{f}_d \neq 0$. To see this, note that $Z^{-1} = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right]$ and $Z^{-1}[1, i] = \sqrt{2}[1, 0]$. We know that f has a symmetrized form, such as $\mathrm{Sym}_n^{n-d}([1, i]; v_1, \ldots, v_d)$. Then up to a scalar factor $2^{n/2}$, $\hat{f} = (Z^{-1})^{\otimes n} f = \mathrm{Sym}_n^{n-d}([1, 0]; u_1, \ldots, u_d)$, where $u_i = Z^{-1} v_i$ for $1 \leq i \leq d$ and u_i and v_i are column vectors in $\mathbb{C}^2$. From this expression for $\hat{f}$, it is clear that all entries of Hamming weight greater than d in $\hat{f}$ are 0. Moreover, if $\hat{f}_d = 0$, then one of the u_i has to be a multiple of $[1, 0]$. This contradicts the degree assumption of f, namely $\mathrm{vd}^+(f) = n - \mathrm{rd}^+(f) = n - d$ and no higher.

In any bipartite graph for $\mathrm{Holant}\left([0,1,0] \mid \hat{f}\right)$, the binary DISEQUALITY $(\neq_2) = [0, 1, 0]$ on the left imposes the condition that half of the edges must take the value 0 and the other half must take the value 1. On the right side, by $f \in \mathscr{V}^+$, we have $d < n/2$, thus $\hat{f}$ requires that less than half of the edges are assigned the value 1. Therefore the Holant is always 0. A similar conclusion was reached in [17] for certain 2-3 bipartite Holant problems with Boolean signatures. However, the importance was not realized at that time.

Under this transformation, one can observe another interesting phenomenon. For any $a, b \in \mathbb{C}$,

$$\operatorname{Holant}([0,1,0] \mid [a,b,1,0,0]) \qquad \text{and} \qquad \operatorname{Holant}([0,1,0] \mid [0,0,1,0,0])$$

take exactly the same value on every signature grid. This is because, to contribute a nonzero term in the Holant, exactly half of the edges must be assigned 1. Then for the first problem, the signature on the right can never contribute a nonzero value involving a or b. Thus the Holant values of these two problems on any signature grid are always the same. Nevertheless, there exists $a, b \in \mathbb{C}$ such that there is no holographic transformation between these two problems. We note that this is the first counter example involving non-unary signatures in the Boolean domain to the converse of the Holant theorem, which provides a negative answer to a conjecture made by Xia in [45] (Conjecture 4.1).

Moreover, $\operatorname{Holant}([0,1,0] \mid [0,0,1,0,0])$ counts the number of Eulerian orientations in a 4-regular graph. This problem was proven #P-hard by Huang and Lu in Theorem V.10 of [30] and plays an important role in our proof of hardness. Translating back to the standard setting, the problem of counting Eulerian orientations in a 4-regular graph is $\operatorname{Holant}([3,0,1,0,3])$. The problem $\operatorname{Holant}([0,1,0] \mid [a,b,1,0,0])$ corresponds to a certain signature $f = Z^{\otimes 4}[a,b,1,0,0]$ of arity 4 with recurrence degree 2. It has a different appearance but induces exactly the same Holant value as the signature for counting Eulerian orientations. Therefore, all such signatures are #P-hard as well. We use this fact later.

For future reference, we also note the following. If $f = g + h$ is of arity n, where $\operatorname{rd}^+(g) = d$, $\operatorname{rd}^-(h) = d'$, and $d + d' < n$, then after a holographic transformation by Z, $\hat{f} = (Z^{-1})^{\otimes n} f$ takes the form $[\widehat{g}_0, \ldots, \widehat{g}_d, 0, \ldots, 0, \widehat{h}_{d'}, \ldots, \widehat{h}_0]$, with $n - d - d' - 1 \geq 0$ zeros in the middle of the signature.

5 Main Result and the Proof of Tractability

Using the definitions from the previous section, we can now formally state our main result.

Theorem 5.1. *Let $\mathcal{F}$ be any set of symmetric, complex-valued signatures in Boolean variables. Then* $\operatorname{Holant}(\mathcal{F})$ *is* #P*-hard unless $\mathcal{F}$ satisfies one of the following conditions, in which case the problem is in* P*:*

1. *All non-degenerate signatures in $\mathcal{F}$ are of arity at most 2;*
2. *$\mathcal{F}$ is $\mathscr{A}$-transformable;*
3. *$\mathcal{F}$ is $\mathscr{P}$-transformable;*
4. *$\mathcal{F} \subseteq \mathscr{V}^\sigma \cup \{f \in \mathscr{R}_2^\sigma \mid \operatorname{arity}(f) = 2\}$ for $\sigma \in \{+,-\}$;*
5. *All non-degenerate signatures in $\mathcal{F}$ are in $\mathscr{R}_2^\sigma$ for $\sigma \in \{+,-\}$.*

Note that any signature in $\mathscr{R}_2^\sigma$ having arity at least 3 is a vanishing signature. Thus all signatures of arity at least 3 in case 5 are vanishing. While both cases 4 and 5 are largely concerned with vanishing signatures, these two cases differ. In case 4, all signatures in $\mathcal{F}$, including unary signatures but excluding binary signatures, must be vanishing of a single type σ; the binary signatures are only required to be in $\mathscr{R}_2^\sigma$. In contrast, case 5 has no requirement placed on degenerate signatures, which include all unary signatures. Then all non-degenerate binary signatures are required to be in $\mathscr{R}_2^\sigma$. Finally all non-degenerate signatures of arity at least 3 are also required to be in $\mathscr{R}_2^\sigma$, which is a strong form of vanishing; they must have a large vanishing degree of type σ.

Case 5 is actually a known tractable case [16, 15]. Every signature (after replacing all degenerate signatures with corresponding ones) is a generalized Fibonacci signature with $m = \sigma 2i$, which means that every signature $[f_0, f_1, \ldots, f_n] \in \mathcal{F}$ satisfies $f_{k+2} = m f_{k+1} + f_k$ for $0 \le k \le n-2$. However, we present a unified proof of tractability based on vanishing signatures, which leads to an alternative algorithm for this case that is essentially the same as one in [15].

Proof of tractability of Theorem 5.1. For any signature grid Ω, Holant_Ω is the product of the Holant on each connected component, so we only need to compute over connected components.

For case 1, after decomposing all degenerate signatures into unary ones, a connected component of the graph is either a path or a cycle and the Holant can be computed using matrix product and trace. Cases 2 and 3 are tractable because, after a particular holographic transformation, their instances are tractable instances of #CSP($\mathcal{F}$) (cf. [11]). For case 4, any binary signature $g \in \mathscr{R}_2^\sigma$ has $\text{rd}^\sigma(g) \le 1$, and thus $\text{vd}^\sigma(g) \ge 1 = \text{arity}(g)/2$. Any signature $f \in \mathscr{V}^\sigma$ has $\text{vd}^\sigma(f) > \text{arity}(f)/2$. If $\mathcal{F}$ contains a signature f of arity at least 3, then it must belong to $\mathscr{V}^\sigma$. Then by the combinatorial view, more than half of the unary signatures are $[1, \sigma i]$, so Holant_Ω vanishes. On the other hand, if the arity of every signature in $\mathcal{F}$ is at most 2, then we have reduced to tractable case 1.

Now consider case 5. First, recursively absorb any unary signature into its neighboring signature. If it is connected to another unary signature, then this produces a global constant factor. If it is connected to a binary signature, then this creates another unary signature. We observe that if $f \in \mathscr{R}_2^\sigma$ has $\text{arity}(f) \ge 2$, then for any unary signature u, after connecting f to u, the signature $\langle f, u \rangle$ still belongs to $\mathscr{R}_2^\sigma$. Hence after recursively absorbing all unary signatures in the above process, we still have a signature grid where all signatures belong to $\mathscr{R}_2^\sigma$. Any remaining signature f that has arity at least 3 belongs to $\mathscr{V}^\sigma$ since $\text{rd}^\sigma(f) \le 1$ and thus $\text{vd}^\sigma(f) \ge \text{arity}(f) - 1 > \text{arity}(f)/2$. Thus we have reduced to tractable case 4. □

6 Dichotomy Theorem for an Arity 4 Signature

Definition 6.1. *A 4-by-4 matrix is* redundant *if its middle two rows and middle two columns are the same. Denote the set of all redundant 4-by-4 matrices over a field $\mathbb{F}$ by* $\text{RM}_4(\mathbb{F})$.

Consider the function $\varphi : \mathbb{C}^{4\times 4} \to \mathbb{C}^{3\times 3}$ defined by

$$\varphi(M) = AMB,$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Intuitively, the operation φ replaces the middle two columns of M with their sum and then the middle two rows of M with their average. (These two steps commute.) Conversely, we have the following function $\psi : \mathbb{C}^{3\times 3} \to \text{RM}_4(\mathbb{C})$ defined by

$$\psi(N) = BNA.$$

Intuitively, the operation ψ duplicates the middle row of N and then splits the middle column evenly into two columns. Notice that $\varphi(\psi(N)) = N$. When restricted to $\text{RM}_4(\mathbb{C})$, φ is an isomorphism between the semi-group of 4-by-4 redundant matrices and the semi-group of 3-by-3 matrices, under matrix multiplication, and ψ is its inverse. To see this, just notice that

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad BA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

are the identity elements of their respective semi-groups.

An example of a redundant matrix is the signature matrix of an arity 4 symmetric signature.

Definition 6.2. *The* signature matrix *of an 4-ary symmetric signature* $f = [f_0, f_1, f_2, f_3, f_4]$ *is*

$$M_f = \begin{bmatrix} f_0 & f_1 & f_1 & f_2 \\ f_1 & f_2 & f_2 & f_3 \\ f_1 & f_2 & f_2 & f_3 \\ f_2 & f_3 & f_3 & f_4 \end{bmatrix}.$$

This definition extends to an asymmetric signature g *as*

$$M_g = \begin{bmatrix} g^{0000} & g^{0010} & g^{0001} & g^{0011} \\ g^{0100} & g^{0110} & g^{0101} & g^{0111} \\ g^{1000} & g^{1010} & g^{1001} & g^{1011} \\ g^{1100} & g^{1110} & g^{1101} & g^{1111} \end{bmatrix}.$$

When we present g *as an* $\mathcal{F}$*-gate, we order the four external edges ABCD counterclockwise. In* M_g*, the row index bits are ordered AB and the column index bits are ordered DC, in a reverse way. This is for convenience so that the signature matrix of the linking of two arity 4* $\mathcal{F}$*-gates is the matrix product of the signature matrices of the two* $\mathcal{F}$*-gates.*

If M_g *is redundant, we also define the* compressed signature matrix *of* g *as* $\widetilde{M_g} = \varphi(M_g)$.

If all signatures in an $\mathcal{F}$-gate have an even arity, then the $\mathcal{F}$-gate also has an even arity. Knowing that binary signatures alone do not produce #P-hardness, with the above constraint in mind, we would like to interpolate other arity 4 signatures using the given arity 4 signature. We are particularly interested in the signature g with signature matrix

$$M_g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{5}$$

the identity element in the semi-group of redundant matrices. Thus, $\widetilde{M_g} = I_3$. Lemma 6.6 shows that the Holant problem with this signature is #P-hard. In Lemma 6.4, we consider when we can interpolate it.

There are three cases in Lemma 6.4 and one of them requires the following technical lemma.

Lemma 6.3. *Let* $M = [B_0 \ B_1 \ \cdots \ B_t]$ *be an* n*-by-*n *block matrix such that there exists* $\lambda \in \mathbb{C}$*, for all integers* $0 \leq k \leq t$*, block* B_k *is an* n*-by-*c_k *matrix for some integer* $c_k \geq 0$*, and the entry of* B_k *at row* r *and column* c *is* $(B_k)_{rc} = r^{c-1}\lambda^{kr}$*, where* $r, c \geq 1$*. If* λ *is nonzero and is not a root of unity, then* M *is nonsingular.*

Proof. We prove by induction on n. If $n = 1$, then the sole entry is λ^k for some nonnegative integer k. This is nonzero since $\lambda \neq 0$. Assume $n > 1$ and let the left-most nonempty block be B_j. We divide row r by λ^{jr}, which is allowed since $\lambda \neq 0$. This effectively changes block B_ℓ into a block of the form $B_{\ell-j}$. Thus, we have another matrix of the same form as M but with a nonempty block B_0. To simplify notation, we also denote this matrix again by M. The first column of B_0 is all 1's. We subtract row $r-1$ from row r, for r from n down to 2. This gives us a new matrix M', and $\det M = \det M'$. Then $\det M'$ is the determinant of the $(n-1)$-by-$(n-1)$ submatrix M'' obtained from M' by removing the first row and column. Now we do column operations (on M'') to return the blocks to the proper form so that we can invoke the induction hypothesis.

For any block B'_k different from B'_0, we prove by induction on the number of columns in B'_k that B'_k can be repaired. In the base case, the rth element of the first column is $(B'_k)_{r1} = \lambda^{kr} - \lambda^{k(r-1)} = \lambda^{k(r-1)}(\lambda^k - 1)$ for $r \geq 2$. We divide this column by $\lambda^k - 1$ to obtain $\lambda^{k(r-1)}$, which is allowed since λ is not a root of unity and $k \neq 0$. This is now the correct form for the rth element of the first column of a block in M''.

Now for the inductive step, assume that the first $d-1$ columns of block B'_k are in the correct form to be a block in M''. That is, for row index $r \geq 2$, which denotes the $(r-1)$-th row of M'', the rth element in the first $d-1$ columns of B'_k have the form $(B'_k)_{rc} = (r-1)^{c-1}\lambda^{k(r-1)}$. The rth element in column d of B'_k currently has the form $(B'_k)_{rd} = r^{d-1}\lambda^{kr} - (r-1)^{d-1}\lambda^{k(r-1)}$. Then we do column operations

$$\begin{aligned}(B'_k)_{rd} - \sum_{c=1}^{d-1}\binom{d-1}{c-1}(B'_k)_{rc} &= r^{d-1}\lambda^{kr} - (r-1)^{d-1}\lambda^{k(r-1)} - \sum_{c=1}^{d-1}\binom{d-1}{c-1}(r-1)^{c-1}\lambda^{k(r-1)} \\ &= r^{d-1}\lambda^{kr} - r^{d-1}\lambda^{k(r-1)} \\ &= r^{d-1}\lambda^{k(r-1)}(\lambda^k - 1)\end{aligned}$$

and divide by $(\lambda^k - 1)$ to get $r^{d-1}\lambda^{k(r-1)}$. Once again, this is allowed since λ is not a root of unity and $k \neq 0$. Then more (of the same) column operations yield

$$r^{d-1}\lambda^{k(r-1)} - \sum_{c=1}^{d-1}\binom{d-1}{c-1}(r-1)^{c-1}\lambda^{k(r-1)} = \lambda^{k(r-1)}\left(r^{d-1} + (r-1)^{d-1} - \sum_{c=1}^{d}\binom{d-1}{c-1}(r-1)^{c-1}\right)$$

and the term in parentheses is precisely $(r-1)^{d-1}$. This gives the correct form for the rth element in column d of B'_k in M''.

Now we repair the columns in B'_0, also by induction on the number of columns. In the base case, if B'_0 only has one column, then there is nothing to prove, since this block has disappeared in M''. Otherwise, $(B'_0)_{r2} = r - (r-1) = 1$, so the second column is already in the correct form to be the first column in M'', and there is still nothing to prove. For the inductive step, assume that columns 2 to $d-1$ are in the correct form to be the first block in M'' for $d \geq 3$. That is, the entry at row $r \geq 2$ and column c from 2 through $d-1$ has the form $(B'_0)_{rc} = (r-1)^{c-2}$. The rth element in column d currently has the form $(B'_0)_{rd} = r^{d-1} - (r-1)^{d-1}$. Then we do the column operations

$$\begin{aligned}(B'_0)_{rd} - \sum_{c=2}^{d-1}\binom{d-1}{c-2}(B'_0)_{rc} &= r^{d-1} - (r-1)^{d-1} - \sum_{c=2}^{d-1}\binom{d-1}{c-2}(r-1)^{c-2} \\ &= (d-1)(r-1)^{d-2}\end{aligned}$$

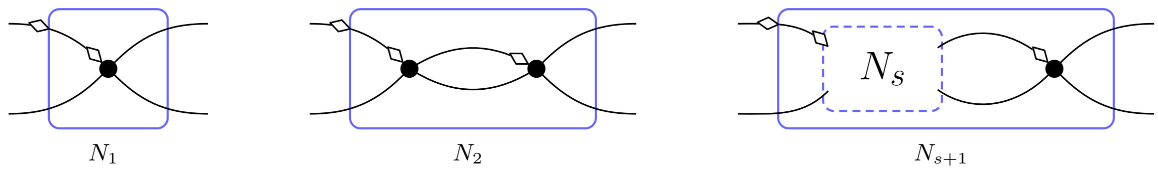


Figure 3: Recursive construction to interpolate g. The vertices are assigned f.

and divide by $d-1$, which is nonzero, to get $(r-1)^{d-2}$. This is the correct form for the rth element in column d of B'_0 in M''. Therefore, we invoke our original induction hypothesis that the $(n-1)$-by-$(n-1)$ matrix M'' has a nonzero determinant, which completes the proof. □

Lemma 6.4. *Let g be the arity 4 signature with M_g given in Equation (5) and let f be an arity 4 signature with complex weights. If M_f is redundant and $\widetilde{M_f}$ is nonsingular, then for any set $\mathcal{F}$ containing f, we have*

$$\operatorname{Holant}(\mathcal{F} \cup \{g\}) \leq_T \operatorname{Holant}(\mathcal{F}).$$

Proof. Consider an instance Ω of $\operatorname{Holant}(\mathcal{F} \cup \{g\})$. Suppose that g appears n times in Ω. We construct from Ω a sequence of instances Ω_s of $\operatorname{Holant}(\mathcal{F})$ indexed by $s \geq 1$. We obtain Ω_s from Ω by replacing each occurrence of g with the gadget N_s in Figure 3 with f assigned to all vertices. In Ω_s, the edge corresponding to the ith significant index bit of N_s connects to the same location as the edge corresponding to the ith significant index bit of g in Ω.

Now to determine the relationship between $\operatorname{Holant}_\Omega$ and $\operatorname{Holant}_{\Omega_s}$, we use the isomorphism between redundant 4-by-4 matrices and 3-by-3 matrices. To obtain Ω_s from Ω, we effectively replace M_g with $M_{N_s} = (M_f)^s$, the sth power of the signature matrix M_f. By the Jordan normal form of $\widetilde{M_f}$, there exists $T, \Lambda \in \mathbb{C}^{3\times 3}$ such that

$$\widetilde{M_f} = T\Lambda T^{-1} = T \begin{bmatrix} \lambda_1 & b_1 & 0 \\ 0 & \lambda_2 & b_2 \\ 0 & 0 & \lambda_3 \end{bmatrix} T^{-1},$$

where $b_1, b_2 \in \{0,1\}$. Note that $\lambda_1\lambda_2\lambda_3 = \det(\widetilde{M_f}) \neq 0$. Also since $\widetilde{M_g} = \varphi(M_g) = I_3$, and $TI_3T^{-1} = I_3$, we have $\psi(T)M_g\psi(T^{-1}) = M_g$. We can view our construction of Ω_s as first replacing each M_g by $\psi(T)M_g\psi(T^{-1})$, which does not change the Holant value, and then replacing each new M_g with $\psi(\Lambda^s) = \psi(\Lambda)^s$ to obtain Ω_s. Observe that

$$\varphi(\psi(T)\psi(\Lambda^s)\psi(T^{-1})) = T\Lambda^s T^{-1} = (\widetilde{M_f})^s = (\varphi(M_f))^s = \varphi((M_f)^s),$$

hence, $\psi(T)\psi(\Lambda^s)\psi(T^{-1}) = M_{N_s}$. (Since $M_g = \psi(T)M_g\psi(T^{-1})$ and $M_{N_s} = \psi(T)\psi(\Lambda^s)\psi(T^{-1})$, replacing each M_g, sandwiched between $\psi(T)$ and $\psi(T^{-1})$, by $\psi(\Lambda^s)$ indeed transforms Ω to Ω_s. We also note that, by the isomorphism, $\psi(T^{-1})$ *is* the multiplicative inverse of $\psi(T)$ within the semi-group of redundant 4-by-4 matrices; but we prefer not to write it as $\psi(T)^{-1}$ since it is not the usual matrix inverse as a 4-by-4 matrix. Indeed, $\psi(T)$ is not invertible as a 4-by-4 matrix.)

In the case analysis below, we stratify the assignments in Ω_s based on the assignment to $\psi(\Lambda^s)$. The inputs to $\psi(\Lambda^s)$ are from $\{0,1\}^2 \times \{0,1\}^2$. However, we can combine the input 01 and 10, since $\psi(\Lambda^s)$ is redundant. Thus we actually stratify the assignments in Ω_s based on the assignment to Λ^s, which takes inputs from $\{0, 1, 2\} \times \{0, 1, 2\}$. In this compressed form, the row and column assignments to Λ^s are the Hamming weight of the two actual binary valued inputs to the uncompressed form $\psi(\Lambda^s)$.

Now we begin the case analysis on the values of b_1 and b_2.

1. Assume $b_1 = b_2 = 0$. We only need to consider the assignments to Λ^s that assign

   - $(0, 0)$ i many times,
   - $(1, 1)$ j many times, and
   - $(2, 2)$ k many times

   since any other assignment contributes a factor of 0. Let c_{ijk} be the sum over all such assignments of the products of evaluations (including the contributions from T and T^{-1}) on Ω_s. Then

   $$\text{Holant}_\Omega = \sum_{i+j+k=n} \frac{c_{ijk}}{2^j}$$

   and the value of the Holant on Ω_s, for $s \geq 1$, is

   $$\text{Holant}_{\Omega_s} = \sum_{i+j+k=n} \left(\lambda_1^i \lambda_2^j \lambda_3^k\right)^s \left(\frac{c_{ijk}}{2^j}\right).$$

   The coefficient matrix is Vandermonde, but it may not have full rank because it may be that $\lambda_1^i \lambda_2^j \lambda_3^k = \lambda_1^{i'} \lambda_2^{j'} \lambda_3^{k'}$ for some $(i, j, k) \neq (i', j', k')$. However, this is not a problem since we are only interested in the sum $\sum c_{ijk}/2^j$. If two coefficients are the same, we replace their corresponding unknowns $\frac{c_{ijk}}{2^j}$ and $\frac{c_{i'j'k'}}{2^{j'}}$ with their sum as a new variable. After all such combinations, we have a Vandermonde system of full rank. In particular, none of the entries are zero since $\lambda_1 \lambda_2 \lambda_3 = \det(\widetilde{M_f}) \neq 0$. Therefore, we can solve the linear system and obtain the value of Holant_Ω.
2. Assume $b_1 \neq b_2$. We can permute the Jordan blocks in Λ so that $b_1 = 1$ and $b_2 = 0$, then $\lambda_1 = \lambda_2$, denoted by λ. We only need to consider the assignments to Λ^s that assign

   - $(0, 0)$ i many times,
   - $(1, 1)$ j many times,
   - $(2, 2)$ k many times, and
   - $(0, 1)$ ℓ many times

   since any other assignment contributes a factor of 0. Let $c_{ijk\ell}$ be the sum over all such assignments of the products of evaluations (including the contributions from T and T^{-1}) on Ω_s. Then

   $$\text{Holant}_\Omega = \sum_{i+j+k=n} \frac{c_{ijk0}}{2^j}$$

   and the value of the Holant on Ω_s, for $s \geq 1$, is

   $$\begin{aligned}\text{Holant}_{\Omega_s} &= \sum_{i+j+k+\ell=n} \lambda^{(i+j)s} \lambda_3^{ks} \left(s\lambda^{s-1}\right)^\ell \left(\frac{c_{ijk\ell}}{2^{j+\ell}}\right) \\ &= \lambda^{ns} \sum_{i+j+k+\ell=n} \left(\frac{\lambda_3}{\lambda}\right)^{ks} s^\ell \left(\frac{c_{ijk\ell}}{\lambda^\ell 2^{j+\ell}}\right).\end{aligned}$$

If λ_3/λ is a root of unity, then take a t such that $(\lambda_3/\lambda)^t = 1$. Then

$$\text{Holant}_{\Omega_{st}} = \lambda^{nst} \sum_{i+j+k+\ell=n} s^\ell \left(\frac{t^\ell c_{ijk\ell}}{\lambda^\ell 2^{j+\ell}} \right)$$

For $s \geq 1$, this gives a coefficient matrix that is Vandermonde. Although this system is not full rank, we can replace all the unknowns $c_{ijk\ell}/2^j$ having $i + j + k = n - \ell$ by their sum to form a new unknown $c'_\ell = \sum_{i+j+k=n-\ell} c_{ijk\ell}/2^j$, where $0 \leq \ell \leq n$. The unknown c'_0 is the Holant of Ω that we seek. The resulting Vandermonde system

$$\text{Holant}_{\Omega_{st}} = \lambda^{nst} \sum_{\ell=0}^{n} s^\ell \left(\frac{t^\ell c'_\ell}{\lambda^\ell 2^\ell} \right)$$

has full rank, so we can solve for the unknowns and obtain the value of $c'_0 = \sum_{i+j+k=n} c_{ijk0}/2^j$.

If λ_3/λ is not a root of unity, then we replace all the unknowns $c_{ijk\ell}/(\lambda^\ell 2^{j+\ell})$ having $i+j = m$ with their sum to form new unknowns $c'_{mk\ell}$, for any $0 \leq m, k, \ell$ and $m + k + \ell = n$. The Holant of Ω is now

$$\text{Holant}_\Omega = \sum_{m+k=n} c'_{mk0}$$

and the value of the Holant on Ω_s is

$$\begin{aligned} \text{Holant}_{\Omega_s} &= \lambda^{ns} \sum_{i+j+k+\ell=n} \left(\frac{\lambda_3}{\lambda} \right)^{ks} s^\ell \left(\frac{c_{ijk\ell}}{\lambda^\ell 2^{j+\ell}} \right) \\ &= \lambda^{ns} \sum_{m+k+\ell=n} \left(\frac{\lambda_3}{\lambda} \right)^{ks} s^\ell c'_{mk\ell}. \end{aligned}$$

After a suitable reordering of the columns, the matrix of coefficients satisfies the hypothesis of Lemma 6.3. Therefore, the linear system has full rank. We can solve for the unknowns and obtain the value of Holant_Ω.

3. Assume $b_1 = b_2 = 1$. In this case, we have $\lambda_1 = \lambda_2 = \lambda_3$, denoted by λ, and we only need to consider the assignments to Λ^s that assign

   - $(0, 0)$ or $(2, 2)$ i many times,
   - $(1, 1)$ j many times,
   - $(0, 1)$ k many times,
   - $(1, 2)$ ℓ many times, and
   - $(0, 2)$ m many times

   since any other assignment contributes a factor of 0. Let $c_{ijk\ell m}$ be the sum over all such assignments of the products of evaluations (including the contributions from T and T^{-1}) on Ω_s. Then

$$\text{Holant}_\Omega = \sum_{i+j=n} \frac{c_{ij000}}{2^j}$$

and the value of the Holant on Ω_s, for $s \geq 1$, is

$$\begin{aligned}\text{Holant}_{\Omega_s} &= \sum_{i+j+k+\ell+m=n} \lambda^{(i+j)s} \left(s\lambda^{s-1}\right)^{k+\ell} \left(s(s-1)\lambda^{s-2}\right)^{m} \left(\frac{c_{ijk\ell m}}{2^{j+k+m}}\right) \\ &= \lambda^{ns} \sum_{i+j+k+\ell+m=n} s^{k+\ell+m}(s-1)^m \left(\frac{c_{ijk\ell m}}{\lambda^{k+\ell+2m}2^{j+k+m}}\right).\end{aligned}$$

We replace all the unknowns $c_{ijk\ell m}/(\lambda^{k+\ell+2m}2^{j+k+m})$ having $i+j=p$ and $k+\ell=q$ with their sum to form new unknowns c'_{pqm}, for any $0 \leq p,q,m$ and $p+q+m=n$. The Holant of Ω is now c'_{n00}. This new linear system is

$$\text{Holant}_{\Omega_s} = \lambda^{ns} \sum_{p+q+m=n} s^{q+m}(s-1)^m c'_{pqm}$$

but is still rank deficient. We now index the columns by (q,m), where $q \geq 0$, $m \geq 0$, and $q+m \leq n$. Correspondingly, we rename the variables $x_{q,m} = c'_{pqm}$. Note that $p = n-q+m$ is determined by (q,m). Observe that the column indexed by (q,m) is the sum of the columns indexed by $(q-1,m)$ and $(q-2,m+1)$ provided $q-2 \geq 0$. Namely, $s^{q+m}(s-1)^m = s^{q-1+m}(s-1)^m + s^{q-2+m+1}(s-1)^{m+1}$. Of course this is only meaningful if $q \geq 2$, $m \geq 0$ and $q+m \leq n$. We write the linear system as

$$\sum_{q\geq 0,\ m\geq 0,\ q+m\leq n} \alpha_{q,m} x_{q,m} = \frac{\text{Holant}_{\Omega_s}}{\lambda^{ns}},$$

where $\alpha_{q,m} = s^{q+m}(s-1)^m$ are the coefficients. Hence $\alpha_{q,m}x_{q,m} = \alpha_{q-1,m}x_{q,m} + \alpha_{q-2,m+1}x_{q,m}$, and we define new variables

$$\begin{aligned} x_{q-1,m} &\leftarrow x_{q,m} + x_{q-1,m} \\ x_{q-2,m+1} &\leftarrow x_{q,m} + x_{q-2,m+1} \end{aligned}$$

from $q = n$ down to 2.

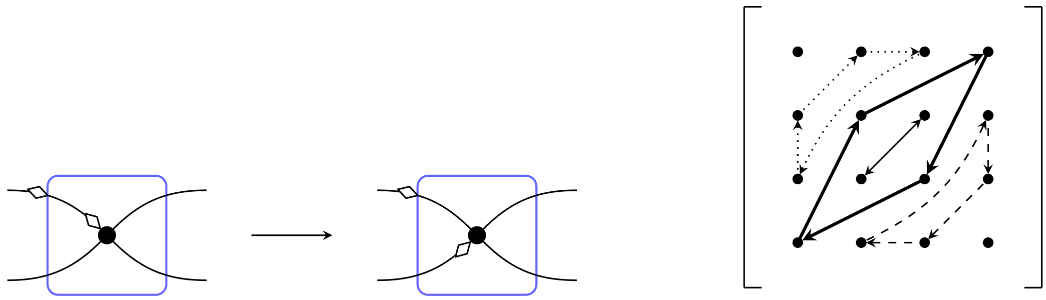
Observe that in each update, the newly defined variables have a decreased index value for q. A more crucial observation is that the column indexed by $(0,0)$ is never updated. This is because, in order to be an updated entry, there must be some $q \geq 2$ and $m \geq 0$ such that $(q-1,m) = (0,0)$ or $(q-2,m+1) = (0,0)$, which is clearly impossible. Hence $x_{0,0} = c'_{n00}$ is still the Holant value on Ω. The $2n+1$ unknowns that remain are

$$x_{0,0},\ x_{1,0},\ x_{0,1},\ x_{1,1},\ x_{0,2},\ x_{1,2},\ \ldots,\ x_{0,n-1},\ x_{1,n-1},\ x_{0,n}$$

and their coefficients in row s are

$$1, s, s(s-1), s^2(s-1), s^2(s-1)^2, \ldots, s^{n-1}(s-1)^{n-1}, s^n(s-1)^{n-1}, s^n(s-1)^n.$$

It is clear that the κ-th entry in this row is a monic polynomial in s of degree κ, where $0 \leq \kappa \leq 2n$, and thus s^κ is a linear combination of the first κ entries. It follows that the coefficient matrix is a product of the standard Vandermonde matrix multiplied to its right by an upper triangular matrix with all 1's on the diagonal. Hence the matrix is nonsingular, and we can solve the linear system, in particular, to compute c'_{n00}. □



(a) A counterclockwise rotation.

(b) Movement of signature matrix entries under a counterclockwise rotation.

Figure 4: The movement of the entries in the signature matrix of an arity 4 signature under a counterclockwise rotation of the input edges. The Hamming weight one entries are in the dotted cycle, the Hamming weight two entries are in the two solid cycles (one has length 4 and the other one is a swap), and the entries of Hamming weight three are in the dashed cycle.

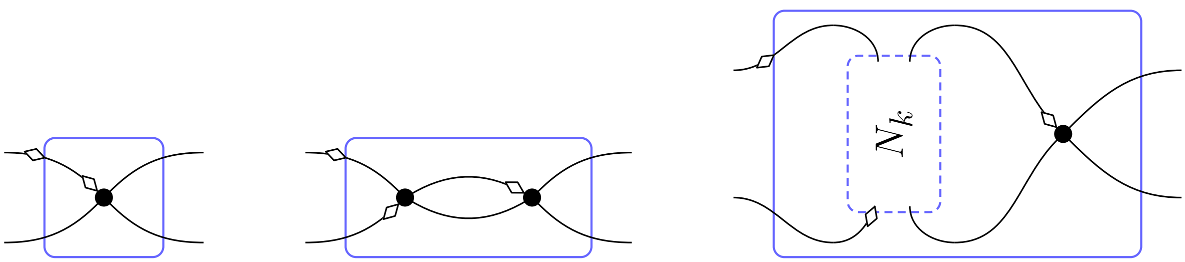


Figure 5: Recursive construction to approximate $[1, 0, 1/3, 0, 1]$. The vertices are assigned g.

For an asymmetric signature, we often want to reorder the input bits under a circular permutation. For a single counterclockwise rotation of 90°, the effect on the entries of the signature matrix of an arity 4 signature is given in Figure 4.

We ultimately derive most of our #P-hardness results through Lemma 6.6. This is done by a reduction from the problem of counting Eulerian orientations on 4-regular graphs, which is the Holant problem $\text{Holant}\left([0, 1, 0] \mid [0, 0, 1, 0, 0]\right)$. Recall that under a holographic transformation by $\left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]$, this bipartite Holant problem becomes the Holant problem $\text{Holant}([1, 0, 1/3, 0, 1])$.

Theorem 6.5 (Theorem V.10 in [30]). Counting-Eulerian-Orientations *is* #P*-hard for 4-regular graphs.*

Lemma 6.6. *Let g be the arity 4 signature with M_g given in Equation (5) so that $\widetilde{M_g} = I_3$. Then* $\text{Holant}(g)$ *is* #P*-hard.*

Proof. We reduce from the Eulerian orientation problem $\text{Holant}(\mathscr{O})$, where $\mathscr{O} = [1, 0, 1/3, 0, 1]$, which is #P-hard by Theorem 6.5. We achieve this via an arbitrarily close approximation using the recursive construction in Figure 5 with g assigned to every vertex.

We claim that the signature matrix M_{N_k} of Gadget N_k is

$$M_{N_k} = \begin{bmatrix} 1 & 0 & 0 & a_k \\ 0 & a_{k+1} & a_{k+1} & 0 \\ 0 & a_{k+1} & a_{k+1} & 0 \\ a_k & 0 & 0 & 1 \end{bmatrix},$$

where $a_k = \frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^k$. This is true for N_0. Inductively assume M_{N_k} has this form. Then the rotated form of the signature matrix for N_k, as described in Figure 4, is

$$\begin{bmatrix} 1 & 0 & 0 & a_{k+1} \\ 0 & a_k & a_{k+1} & 0 \\ 0 & a_{k+1} & a_k & 0 \\ a_{k+1} & 0 & 0 & 1 \end{bmatrix}. \tag{6}$$

The action of g on the far right side of N_{k+1} is to replace each of the middle two entries in the middle two rows of the matrix in (6) with their average, $(a_k + a_{k+1})/2 = a_{k+2}$. This gives $M_{N_{k+1}}$.

Let G be a graph with n vertices and $H_{\mathscr{O}}$ (resp. H_{N_k}) be the Holant value on G with all vertices assigned $\mathscr{O}$ (resp. N_k). Since each signature entry in $\mathscr{O}$ can be expressed as a rational number with denominator 3, each term in the sum of $H_{\mathscr{O}}$ can be expressed as a rational number with denominator 3^n, and $H_{\mathscr{O}}$ itself is a sum of 2^{2n} such terms, where $2n$ is the number of edges in G. If the error $|H_{N_k} - H_{\mathscr{O}}|$ is at most $1/3^{n+1}$, then we can recover $H_{\mathscr{O}}$ from H_{N_k} by selecting the nearest rational number to H_{N_k} with denominator 3^n.

For each signature entry x in $M_{\mathscr{O}}$, its corresponding entry $\tilde{x}$ in M_{N_k} satisfies $|\tilde{x} - x| \le x/2^k$. Then for each term t in the Holant sum $H_{\mathscr{O}}$, its corresponding term $\tilde{t}$ in the sum H_{N_k} satisfies $t(1 - 1/2^k)^n \le \tilde{t} \le t(1 + 1/2^k)^n$, thus $-t(1 - (1 - 1/2^k)^n) \le \tilde{t} - t \le t((1 + 1/2^k)^n - 1)$. Since $1 - (1 - 1/2^k)^n \le (1 + 1/2^k)^n - 1$, we get $|\tilde{t} - t| \le t((1 + 1/2^k)^n - 1)$. Also each term $t \le 1$. Hence

$$|H_{N_k} - H_{\mathscr{O}}| \le 2^{2n}((1 + 1/2^k)^n - 1) < 1/3^{n+1},$$

if we take $k = 4n$. □

We summarize our progress with the following corollary, which combines Lemmas 6.4 and 6.6.

Corollary 6.7. *Let f be an arity 4 signature with complex weights. If M_f is redundant and $\widetilde{M_f}$ is nonsingular, then* Holant(f) *is* #P*-hard.*

In order to make Corollary 6.7 more applicable, we show that for an arity 4 signature f, the redundancy of M_f and the nonsingularity of $\widetilde{M_f}$ are invariant under an invertible holographic transformation.

Lemma 6.8. *Let f be an arity 4 signature with complex weights, $T \in \mathbb{C}^{2\times 2}$ a matrix, and $\hat{f} = T^{\otimes 4} f$. If M_f is redundant, then $M_{\hat{f}}$ is also redundant and* $\det(\varphi(M_{\hat{f}})) = \det(\varphi(M_f)) \det(T)^6$.

Proof. Since $\hat{f} = T^{\otimes 4} f$, we can express $M_{\hat{f}}$ in terms of M_f and T as

$$M_{\hat{f}} = T^{\otimes 2} M_f \left(T^{\mathsf{T}}\right)^{\otimes 2}. \tag{7}$$

This can be directly checked. Alternatively, this relation is known (and can also be directly checked) had we not introduced the flip of the middle two columns, i.e., if the columns were ordered

00, 01, 10, 11 by the last two bits in f and $\hat{f}$. Instead, the columns are ordered by 00, 10, 01, 11 in M_f and $M_{\hat{f}}$. Let $T = (t^i_j)$, where row index i and column index j range from $\{0, 1\}$. Then $T^{\otimes 2} = (t^i_j t^{i'}_{j'})$, with row index ii' and column index jj'. Let

$$\mathcal{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

then $\mathcal{E}T^{\otimes 2}\mathcal{E} = T^{\otimes 2}$, i.e., a simultaneous row flip $ii' \leftrightarrow i'i$ and column flip $jj' \leftrightarrow j'j$ keep $T^{\otimes 2}$ unchanged. Then the known relations $M_{\hat{f}}\mathcal{E} = T^{\otimes 2} M_f \mathcal{E} \left(T^{\mathsf{T}}\right)^{\otimes 2}$ and $\mathcal{E}\left(T^{\mathsf{T}}\right)^{\otimes 2}\mathcal{E} = \left(T^{\mathsf{T}}\right)^{\otimes 2}$ imply Equation (7).

Now $X \in \mathrm{RM}_4(\mathbb{C})$ iff $\mathcal{E}X = X = X\mathcal{E}$. Then it follows that $M_{\hat{f}} \in \mathrm{RM}_4(\mathbb{C})$ if $M_f \in \mathrm{RM}_4(\mathbb{C})$. For the two matrices A and B in the definition of φ, we note that $BA = M_g$, where M_g given in Equation (5) is the identity element of the semi-group $\mathrm{RM}_4(\mathbb{C})$. Since $M_f \in \mathrm{RM}_4(\mathbb{C})$, we have $BAM_f = M_f = M_f BA$. Then we have

$$\begin{aligned} \varphi(M_{\hat{f}}) &= AM_{\hat{f}}B = A\left(T^{\otimes 2} M_f \left(T^{\mathsf{T}}\right)^{\otimes 2}\right)B \\ &= (AT^{\otimes 2}B)(AM_fB)(A\left(T^{\mathsf{T}}\right)^{\otimes 2}B) \\ &= \varphi(T^{\otimes 2})\varphi(M_f)\varphi(\left(T^{\mathsf{T}}\right)^{\otimes 2}). \end{aligned} \tag{8}$$

Another direct calculation shows that

$$\det(\varphi(T^{\otimes 2})) = \det(T)^3 = \det(\varphi(\left(T^{\mathsf{T}}\right)^{\otimes 2})).$$

Thus, by applying determinant to both sides of Equation (8), we have

$$\det(\varphi(M_{\hat{f}})) = \det(\varphi(M_f))\det(T)^6$$

as claimed. □

In particular, for a nonsingular matrix $T \in \mathbb{C}^{2\times 2}$, M_f is redundant and $\widetilde{M_f}$ is nonsingular iff $M_{\hat{f}}$ is redundant and $\widetilde{M_{\hat{f}}}$ is nonsingular. From Corollary 6.7 and Lemma 6.8 we have the following corollary.

Corollary 6.9. *Let f be an arity 4 signature with complex weights. If there exists a nonsingular matrix $T \in \mathbb{C}^{2\times 2}$ such that $\hat{f} = T^{\otimes 4} f$, where $M_{\hat{f}}$ is redundant and $\widetilde{M_{\hat{f}}}$ is nonsingular, then* $\mathrm{Holant}(f)$ *is* #P*-hard.*

The following lemma applies Corollary 6.7.

Lemma 6.10. *Let $f_k = ck\alpha^{k-1} + d\alpha^k$, where $c \neq 0$ and $0 \leq k \leq 4$. Then the problem* $\mathrm{Holant}([f_0, f_1, f_2, f_3, f_4])$ *is* #P*-hard unless $\alpha = \pm i$, in which case the Holant vanishes.*

Proof. If $\alpha = \pm i$, then $\mathrm{rd}^{\pm}(f) = 1$, $\mathrm{vd}^{\pm}(f) = 3$, and so $f = [f_0, f_1, f_2, f_3, f_4]$ is vanishing by Theorem 4.13. Otherwise, a holographic transformation with orthogonal basis $T = \frac{1}{\sqrt{1+\alpha^2}}\left[\begin{smallmatrix} 1 & \alpha \\ \alpha & -1 \end{smallmatrix}\right]$ transforms f to $\hat{f} = [t, 1, 0, 0, 0]$ for some $t \in \mathbb{C}$ after normalizing the second entry. (See Appendix B

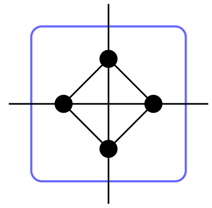


Figure 6: The tetrahedron gadget. Each vertex is assigned $\hat{f} = [t, 1, 0, 0, 0]$.

for details.) Using the tetrahedron gadget in Figure 6 with $\hat{f}$ assigned to each vertex, we have a gadget with signature

$$h = [t^4 + 6t^2 + 3, t^3 + 3t, t^2 + 1, t, 1].$$

Since the determinant of $\widetilde{M_h}$ is 4, the compressed signature matrix of this gadget is nonsingular, so we are done by Corollary 6.7. □

Now we are ready to prove a dichotomy for a single arity 4 signature.

Theorem 6.11. *If f is a non-degenerate, symmetric, complex-valued signature of arity 4 in Boolean variables, then* Holant(f) *is* #P*-hard unless f is $\mathscr{A}$-transformable, $\mathscr{P}$-transformable, or vanishing, in which case the problem is in* P*.*

Proof. Let $f = [f_0, f_1, f_2, f_3, f_4]$. If the compressed signature matrix $\widetilde{M_f}$ is nonsingular, then Holant(f) is #P-hard by Corollary 6.7, so assume that the rank of $\widetilde{M_f}$ is at most 2. Then we have

$$a \begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix} + 2b \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} + c \begin{pmatrix} f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

for some $a, b, c \in \mathbb{C}$ that are not all zero. If $a = c = 0$, then $b \neq 0$, so $f_1 = f_2 = f_3 = 0$. In this case, $f \in \mathscr{P}$ is a generalized equality signature, so f is $\mathscr{P}$-transformable. Now suppose a and c are not both 0. Then f satisfies a second order recurrence relation. If the roots of the characteristic polynomial of the recurrence relation are distinct, then $f_k = \alpha_1^{4-k}\alpha_2^k + \beta_1^{4-k}\beta_2^k$, where $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$. A holographic transformation by $\left[\begin{smallmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{smallmatrix}\right]$ transforms f to $=_4$ and we can use Theorem 2.9′ to show that f is either $\mathscr{A}$- or $\mathscr{P}$-transformable. Otherwise, the characteristic polynomial has a double root α and there are two cases. In the first, for any $0 \leq k \leq 2$, $f_k = ck\alpha^{k-1} + d\alpha^k$, where $c \neq 0$. In the second, for any $0 \leq k \leq 2$, $f_k = c(4-k)\alpha^3 + d\alpha^{4-k}$, where $c \neq 0$. These cases map between each other under a holographic transformation by $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, so assume that we are in the first case. Then we are done by Lemma 6.10. □

The next lemma is related to vanishing signatures. It appears here because its proof uses similar techniques to those in this section.

Lemma 6.12. *If $f = [0, 1, 0, \ldots, 0]$ and $g = [0, \ldots, 0, 1, 0]$ are both of arity $n \geq 3$, then the problem* Holant $([0, 1, 0] \mid \{f, g\})$ *is* #P*-hard.*

(a) The circle is assigned f, the triangle is assigned g, and the squares are assigned $\neq_2$.

(b) The circle is assigned h', the triangle is assigned h'', and the squares are assigned $\neq_2$.

Figure 7: Gadget constructions to obtain redundant arity 4 signatures.

Proof. Our goal is to obtain a signature that satisfies the hypothesis of Corollary 6.9.

The gadget in Figure 7a, with f assigned to the circle vertex, g assigned to the triangle vertex, and $\neq_2$ assigned to the square vertices, has signature h with signature matrix

$$M_h = \begin{bmatrix} 0 & 0 & 0 & v \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $v = n-2$ is positive since $n \geq 3$. Although this signature matrix is redundant, its compressed form is singular. Rotating this gadget 90° clockwise and 90° counterclockwise yield signatures h' and h'' respectively, with signature matrices

$$M_{h'} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & v & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad M_{h''} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & v & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The gadget in Figure 7b, with h' assigned to the circle vertex, h'' assigned to the triangle vertex, and $\neq_2$ assigned to the square vertices, has a signature r with signature matrix

$$M_r = M_{h'} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} M_{h''} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & v & v^2+1 & 0 \\ 0 & 1 & v & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Note that the effect of the $\neq_2$ signatures is to reverse all four rows of $M_{h''}$ before multiplying it to the right of $M_{h'}$. Although this signature matrix is not redundant, every entry of Hamming weight two is nonzero since v is positive.

Now we claim that we can use r to interpolate the following signature r', for any nonzero value $t \in \mathbb{C}$, via the construction in Figure 8. Define $p^{\pm} = (v \pm \sqrt{v^2+4})/2$, $P = \left[\begin{smallmatrix} 1 & 1 \\ p^+ & p^- \end{smallmatrix}\right]$, and $T = P \left[\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right] P^{-1}$ where $t \in \mathbb{C}$ is any nonzero value. Then the signature matrix of r' is

$$M_{r'} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{T}} & 0 \\ 0 & & & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{9}$$

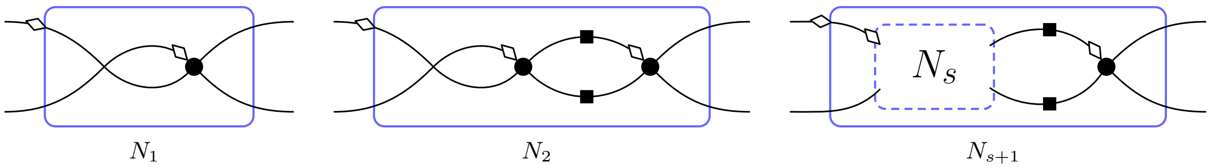


Figure 8: Recursive construction to interpolate a signature r' that is only a rotation away from having a redundant signature matrix and nonsingular compressed matrix. The circles are assigned r and the squares are assigned $\neq_2$.

Consider an instance Ω of $\operatorname{Holant}(\neq_2 \mid \mathcal{F} \cup \{r'\})$. Suppose that r' appears n times in Ω. We construct from Ω a sequence of instances Ω_s of $\operatorname{Holant}(\neq_2 \mid \mathcal{F})$ indexed by $s \geq 1$. We obtain Ω_s from Ω by replacing each occurrence of r' with the gadget N_s in Figure 8 with r assigned to the circle vertices and $\neq_2$ assigned to the square vertices. In Ω_s, the edge corresponding to the ith significant index bit of N_s connects to the same location as the edge corresponding to the ith significant index bit of g in Ω.

The signature matrix of N_s is the sth power of the matrix obtained from M_r after reversing all rows, and then switching the first and last rows of the final product, namely

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & v & 0 \\ 0 & v & v^2+1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^s = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & v & 0 \\ 0 & v & v^2+1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & v & 0 \\ 0 & v & v^2+1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{s-1}.$$

The twist of the two input edges on the left side for the first copy of M_r switches the middle two rows, which is equivalent to a total reversal of all rows, followed by the switching of the first and last rows. The total reversals of rows for all subsequent $s-1$ copies of M_r are due to the presence of $\neq_2$ signatures.

After such reversals of rows, it is clear that the matrix is a direct sum of block matrices indexed by $\{00, 11\} \times \{00, 11\}$ and $\{01, 10\} \times \{10, 01\}$. Furthermore, in the final product, the block indexed by $\{00, 11\} \times \{00, 11\}$ is $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Thus in the gadget N_s, the only entries of M_{N_s} that vary with s are the four entries in the middle. These middle four entries of M_{N_s} form the 2-by-2 matrix $\left[\begin{smallmatrix} 1 & v \\ v & v^2+1 \end{smallmatrix}\right]^s$. Since $\left[\begin{smallmatrix} 1 & v \\ v & v^2+1 \end{smallmatrix}\right] = P \left[\begin{smallmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{smallmatrix}\right] P^{-1}$, where $\lambda_\pm = (v^2 + 2 \pm v\sqrt{v^2+4})/2$ are the eigenvalues, we have

$$\left[\begin{smallmatrix} 1 & v \\ v & v^2+1 \end{smallmatrix}\right]^s = P \left[\begin{smallmatrix} \lambda_+^s & 0 \\ 0 & \lambda_-^s \end{smallmatrix}\right] P^{-1}.$$

The determinant is $\lambda_+\lambda_- = 1$, so the eigenvalues are nonzero. Since v is positive, the ratio of the eigenvalues λ_+/λ_- is not a root of unity, so neither λ_+ nor λ_- is a root of unity.

Now we determine the relationship between $\operatorname{Holant}_\Omega$ and $\operatorname{Holant}_{\Omega_s}$. We can view our construction of Ω_s as first replacing $M_{r'}$ with

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{P}} & 0 \\ 0 & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & t & 0 & 0 \\ 0 & 0 & t^{-1} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{P^{-1}}} & 0 \\ 0 & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

which does not change the Holant value, and then replacing the new signature matrix in the middle with the signature matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & \lambda_+^s & 0 & 0 \\ 0 & 0 & \lambda_-^s & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

We stratify the assignments in Ω based on the assignments to the n occurrences of the signature matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & t & 0 & 0 \\ 0 & 0 & t^{-1} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The inputs to this matrix are from $\{0,1\}^2 \times \{0,1\}^2$, which correspond to the four input bits. Recall the way rows and columns of a signature matrix are ordered from Definition 6.2. Thus, e.g., the entry t corresponds to the cyclic input bit pattern 0110 in counterclockwise order. We only need to consider the assignments that assign

- i many times the bit pattern 0110,
- j many times the bit pattern 1001, and
- k many times the bit patterns 0011 or 1100,

since any other assignment contributes a factor of 0. Let c_{ijk} be the sum over all such assignments of the product of evaluations (including the contributions from the block matrices containing P and P^{-1}) on Ω. Then

$$\text{Holant}_\Omega = \sum_{i+j+k=n} t^{i-j} c_{ijk}$$

and the value of the Holant on Ω_s, for $s \geq 1$, is

$$\text{Holant}_{\Omega_s} = \sum_{i+j+k=n} \lambda_+^{si} \lambda_-^{sj} c_{ijk} = \sum_{i+j+k=n} \lambda_+^{s(i-j)} c_{ijk}.$$

This Vandermonde system does not have full rank. However, we can define for $-n \leq \ell \leq n$,

$$c'_\ell = \sum_{\substack{i-j=\ell \\ i+j+k=n}} c_{ijk}.$$

Then the Holant of Ω is

$$\text{Holant}_\Omega = \sum_{-n \leq \ell \leq n} t^\ell c'_\ell$$

and the Holant of Ω_s is

$$\text{Holant}_{\Omega_s} = \sum_{-n \leq \ell \leq n} \lambda_+^{s\ell} c'_\ell.$$

Now this Vandermonde has full rank because λ_+ is neither zero nor a root of unity. Therefore, we can solve for the unknowns $\{c'_\ell\}$ and obtain the value of Holant_Ω. This completes our claim that we can interpolate the signature r' in Equation (9), for any nonzero $t \in \mathbb{C}$.

Let $t = (\sqrt{v^2+8} + \sqrt{v^2+4})/2$ so $t^{-1} = (\sqrt{v^2+8} - \sqrt{v^2+4})/2$. Let $a = (\sqrt{v^2+8} - v)/2$ and $b = (\sqrt{v^2+8} + v)/2$, then $ab = 2$ and both a and b are nonzero. One can verify that

$$P \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} P^{-1} = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}.$$

Thus, the signature matrix for r' is

$$M_{r'} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & a & 1 & 0 \\ 0 & 1 & b & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

After a counterclockwise rotation of $90°$ on the edges of r', we have a signature r'' with a redundant signature matrix

$$M_{r''} = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ b & 0 & 0 & 0 \end{bmatrix}.$$

Its compressed signature matrix

$$\widetilde{M_{r''}} = \begin{bmatrix} 0 & 0 & a \\ 0 & 2 & 0 \\ b & 0 & 0 \end{bmatrix}$$

is nonsingular. After a holographic transformation by $Z^{-1} = \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]^{-1}$, the binary disequality $(\neq_2) = [0,1,0]$ is transformed to the binary equality $(=_2) = [1,0,1]$. Thus $\operatorname{Holant}([0,1,0] \mid r'')$ is transformed to $\operatorname{Holant}\left(=_2 \mid Z^{\otimes 4} r''\right)$, which is the same as $\operatorname{Holant}(Z^{\otimes 4} r'')$. We conclude that this Holant problem is #P-hard by Corollary 6.9. □

7 Vanishing Signatures Revisited

With Corollary 6.7, Corollary 6.9, and Lemma 6.12 in hand, we revisit the vanishing signatures to determine what signatures combine with them to give #P-hardness. We begin with unary signatures and their tensor powers.

Lemma 7.1. *Let $f \in \mathscr{V}^\sigma$ be a symmetric signature with $\operatorname{rd}^\sigma(f) \geq 2$ where $\sigma \in \{+,-\}$. Suppose the signature v is a tensor power of a unary signature u. If u is not a multiple of $[1, \sigma i]$, then* $\operatorname{Holant}(\{f, v\})$ *is* #P*-hard.*

Proof. We consider $\sigma = +$ since the other case is similar. Since $f \in \mathscr{V}^+$, we have $\operatorname{arity}(f) > 2\operatorname{rd}^+(f) \geq 4$, and $\operatorname{vd}^+(f) > 0$. As $\operatorname{rd}^+(f) \geq 2$, f is a nonzero signature. By Lemma 4.15, with zero or more self loops of f, we can construct some f' with $\operatorname{rd}^+(f') = 2$ and arity $n \geq 5$. We can repeatedly apply Lemma 4.15, since in each step we reduce the recurrence degree rd^+ by exactly one, which remains positive and thus the signature is nonzero. Being obtained from f by self loops, it remains in $\mathscr{V}^+$. The process can be continued. After two more self loops, we have $[1,i]^{\otimes(n-4)}$. Assume $v = u^{\otimes n'}$.

Now we have two degenerate signatures and we can connect one to the other to get a tensor power of a smaller positive arity as long as their arities are not the same. This procedure is like

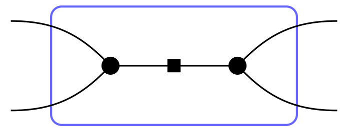


Figure 9: The circles are assigned $[t, 1, 0, 0]$ and the square is assigned $\neq_2$.

the subtractive Euclidean algorithm, which halts when the two arities are equal, and that would be $t = \gcd(n-4, n')$. Alternatively, there are integers x and y such that $xn' + y(n-4) = t$, by replacing x by $x + z(n-4)$ and y by $y - zn'$, for any integer z, we may assume $x > 0$ and $y < 0$. Then if we connect $|y|$ copies of $[1,i]^{\otimes(n-4)}$ to x copies of $v = u^{\otimes n'}$, we can realize $u^{\otimes t}$ (note that u is not any multiple of $[1,i]$ and thus $\langle u, [1,i]\rangle$ is a nonzero constant). We can realize $g = u^{\otimes(n-4)}$ by putting $(n-4)/t$ many copies of $u^{\otimes t}$ together.

Now connect this g back to f'. Since the unary u is not any multiple of $[1,i]$, we can directly verify that $g \notin \mathscr{R}^+_{n-4}$ and thus $\text{rd}^+(g) = \text{arity}(g) = n-4$, and $\text{vd}^+(g) = 0$. By Lemma 4.14, we get $f'' = \langle f', g\rangle$ of arity 4 and $\text{rd}^+(f'') = 2$. One can verify that $\text{Holant}(f'')$ is #P-hard by Corollary 6.7, by writing $f''_k = i^k p(k)$ for some polynomial p of degree exactly 2. A more revealing proof of the #P-hardness of $\text{Holant}(f'')$ is by noticing that this is the problem $\text{Holant}\left(=_2 \mid f''\right)$, which is equivalent to $\text{Holant}\left(\neq_2 \mid \widehat{f''}\right)$ under the holographic transformation $Z = \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]$. By $\text{rd}^+(f'') = 2$, $\widehat{f''}$ takes the form $[\widehat{f''}_0, \widehat{f''}_1, \widehat{f''}_2, 0, 0]$, with $\widehat{f''}_2 \neq 0$. Then $\text{Holant}\left(\neq_2 \mid [\widehat{f''}_0, \widehat{f''}_1, \widehat{f''}_2, 0, 0]\right) \equiv \text{Holant}\left(\neq_2 \mid [0,0,1,0,0]\right)$, the Eulerian Orientation problem (see Section 4.3), which is #P-hard by Theorem 6.5. ☐

Next we consider binary signatures.

Lemma 7.2. *Let $f \in \mathscr{V}^\sigma$ be a symmetric non-degenerate signature where $\sigma \in \{+,-\}$. If $g \notin \mathscr{R}^\sigma_2$ is a non-degenerate binary signature, then* $\text{Holant}(\{f, g\})$ *is* #P-*hard.*

Proof. We consider $\sigma = +$ since the other case is similar. A unary signature is degenerate. If a binary symmetric signature f is vanishing, then its vanishing degree is greater than 1, hence at least 2, and therefore f is also degenerate. Since we assume f is non-degenerate, $\text{arity}(f) \geq 3$.

We prove the lemma by induction on the arity of f. There are two base cases, $\text{arity}(f) = 3$ and $\text{arity}(f) = 4$. However, the arity 3 case is easily reduced to the arity 4 case. We show this first, and then show that the lemma holds for the arity 4 case.

Assume $\text{arity}(f) = 3$. Since $f \in \mathscr{V}^+$, we have $\text{rd}^+(f) < 3/2$, thus $f \in \mathscr{R}^+_2$. From $\text{rd}^+(f) \leq 1$ we have $\text{vd}^+(f) \geq 2$. On the other hand, f is non-degenerate, and so $\text{vd}^+(f) < 3$, thus $\text{vd}^+(f) = 2$.

We connect two copies of f together by one edge to get an arity 4 signature f'. By the geometric construction, this may not appear to be a symmetric signature, but we show that f' is in fact *symmetric*, non-degenerate and vanishing. It is clearly a vanishing signature, since f is vanishing. Consider the Z transformation, under which f is transformed into $\hat{f} = [t, 1, 0, 0]$ for some t up to a nonzero constant. The $=_2$ on the connecting edge between the two copies of f is transformed into $\neq_2$. In the bipartite setting, our construction is the same as the gadget in Figure 9. One can verify that the resulting signature is $\hat{f}' = [2t, 1, 0, 0, 0]$. The crucial observation is that it takes the same value 0 on inputs 1010 and 1100, where the left two bits are input to one copy of f and the right two bits are for another. The corresponding signature f' is non-degenerate, with $\text{rd}^+(f') = 1$ and vanishing.

Next we consider the base case of $\text{arity}(f) = 4$. Since $f \in \mathscr{V}^+$, we have $\text{vd}^+(f) > 2$ and $\text{rd}^+(f) < 2$. Since f is non-degenerate we have $\text{rd}^+(f) \neq -1, 0$, hence $\text{rd}^+(f) = 1$ and $\text{vd}^+(f) = 3$. Also by assumption, the given binary $g \notin \mathscr{R}_2^+$, we have $\text{rd}^+(g) = 2$. Once again, consider the holographic transformation by Z. This gives

$$\begin{aligned}\text{Holant}\,(=_2 \mid \{f, g\}) &\equiv_T \text{Holant}\left([1,0,1]Z^{\otimes 2} \mid \{(Z^{-1})^{\otimes 4} f, (Z^{-1})^{\otimes 2} g\}\right)\\ &\equiv_T \text{Holant}\left([0,1,0] \mid \{\hat{f}, \hat{g}\}\right),\end{aligned}$$

where up to a nonzero constant, $\hat{f} = [t, 1, 0, 0, 0]$ and $\hat{g} = [a, b, 1]$, for some $t, a, b \in \mathbb{C}$. We have $a - b^2 \neq 0$ since g is non-degenerate.

Our next goal is to show that we can realize a signature of the form $[c, 0, 1]$ where $c \neq 0$. If $b = 0$, then $\hat{g}$ is what we want since in this case $a = a - b^2 \neq 0$. Now we assume $b \neq 0$.

Connecting $\hat{g}$ to $\hat{f}$ via $\neq_2$, we get $[t+2b, 1, 0]$. If $t \neq -2b$, then by Lemma A.1, we can interpolate any binary signature of the form $[v, 1, 0]$. Otherwise $t = -2b$. Then we connect two copies of $\hat{g}$ via $\neq_2$, and get $\widehat{g'} = [2ab, a + b^2, 2b]$. Connecting this $\widehat{g'}$ to $\hat{f}$ via $\neq_2$, we get $[2(a - b^2), 2b, 0]$, using $t = -2b$. Since $a \neq b^2$ and $b \neq 0$, we can interpolate any $[v, 1, 0]$ again by Lemma A.1.

Hence, we have the signature $[v, 1, 0]$, where $v \in \mathbb{C}$ is for us to choose. We construct the gadget in Figure 10 with the circles assigned $[v, 1, 0]$, the squares assigned $\neq_2$, and the triangle assigned $[a, b, 1]$. The resulting gadget has signature $[a + 2bv + v^2, b + v, 1]$, which can be verified by the matrix product

$$\begin{bmatrix} v & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ b & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a + 2bv + v^2 & b + v \\ b + v & 1 \end{bmatrix}.$$

By setting $v = -b$, we get $[c, 0, 1]$, where $c = a - b^2 \neq 0$.

With this signature $[c, 0, 1]$, we construct the gadget in Figure 11, where $[c, 0, 1]$ is assigned to the circle vertex of arity two in Figure 11b and $\hat{f}$ is assigned to the four circle vertices of arity four in Figure 11a. We get a signature

$$\hat{h} = [3c^2 + 6ct^2 + t^4, 3ct + t^3, c + t^2, t, 1].$$

We note that this computation is reminiscent of matchgate signatures. The internal edge function $[1, 0, c]$ (which is a flip from $[c, 0, 1]$ since both sides are connected to $\neq_2$) is a generalized equality signature, and the signature $\hat{f}$ on the four circle vertices is a weighted version of the matching function At-Most-One.

The compressed signature matrix of $\hat{h}$ is

$$\widetilde{M_{\hat{h}}} = \begin{bmatrix} 3c^2 + 6ct^2 + t^4 & 2(3ct + t^3) & c + t^2 \\ 3ct + t^3 & 2(c + t^2) & t \\ c + t^2 & 2t & 1 \end{bmatrix}$$

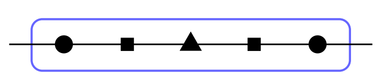
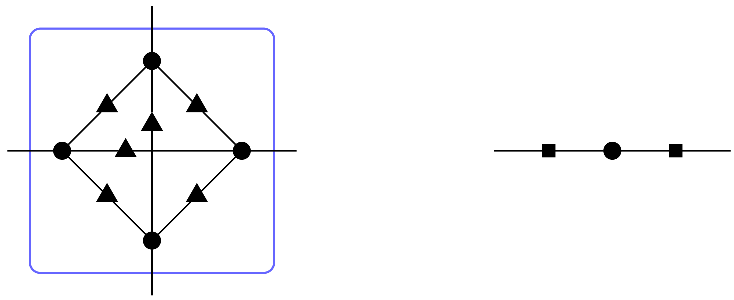


Figure 10: A sequence of binary gadgets that forms another binary gadget. The circles are assigned $[v, 1, 0]$, the squares are assigned $\neq_2$, and the triangle is assigned $[a, b, 1]$.



(a) The tetrahedron gadget with edge signatures given in (b).

(b) The gadget representing an edge labeled by a triangle in (a).

Figure 11: The tetrahedron gadget with each triangle replaced by the edge in (b), where the circle is assigned $[c, 0, 1]$ and the squares are assigned $\neq_2$. The four circles in (a) are assigned $[t, 1, 0, 0, 0]$.

and its determinant is $4c^3 \neq 0$. Thus $\widetilde{M_{\hat{h}}}$ is nonsingular. After a holographic transformation by $T = \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]$, the binary disequality $(\neq_2) = [0, 1, 0]$ is transformed to the binary equality $(=_2) = [1, 0, 1]$. Thus $\operatorname{Holant}\left([0,1,0] \mid \hat{h}\right)$ is transformed to $\operatorname{Holant}\left(=_2 \mid T^{\otimes 4}\hat{h}\right)$, which is the same as $\operatorname{Holant}(T^{\otimes 4}\hat{h})$. Then we are done by Corollary 6.9.

Now we do the induction step. Assume f is of arity $n \geq 5$. Since f is non-degenerate, $\operatorname{rd}^+(f) \neq -1, 0$. First suppose $\operatorname{rd}^+(f) = 1$, then connect the binary g to f to get $f' = \langle f, g \rangle$. We have noted that $\operatorname{rd}^+(g) = 2$, then $\operatorname{vd}^+(g) = 0$. By Lemma 4.14 we know that $\operatorname{rd}^+(f') = 1$ and $\operatorname{arity}(f') = n - 2 \geq 3$. Thus f' is vanishing. Also f' is non-degenerate, for otherwise let $f' = [a, b]^{\otimes(n-2)}$. If $[a, b]$ is a multiple of $[1, i]$, then $\operatorname{rd}^+(f') \leq 0$, which is false. If $[a, b]$ is not a multiple of $[1, i]$, then it can be directly checked that $f' \notin \mathscr{R}^+_{n-2}$, and $\operatorname{rd}^+(f') = n - 2 > 1$, which is also false. Hence f' is a non-degenerate vanishing signature of arity $n - 2$. By induction hypothesis we are done.

We now suppose $\operatorname{rd}^+(f) = t \geq 2$. Since f is non-degenerate it is certainly nonzero. Since it is vanishing, certainly $\operatorname{vd}^+(f) > 0$. Hence we may apply Lemma 4.15. Let f' by obtained from f by a self loop, then $\operatorname{rd}^+(f') = t - 1 \geq 1$ and $\operatorname{arity}(f') = n - 2$. Clearly f' is still vanishing. We claim that f' is non-degenerate. This is proved by the same argument as above. If f' were degenerate, then either $\operatorname{rd}^+(f') \leq 0$ or $\operatorname{rd}^+(f') = \operatorname{arity}(f')$ which would contradict f' being a vanishing signature. Therefore, we can apply the induction hypothesis. This finishes our proof. □

Finally, we consider a pair of vanishing signatures of opposite type, both of arity at least 3. We show that opposite types of vanishing signatures cannot mix. More formally, vanishing signatures of opposite types, when put together, lead to #P-hardness.

Lemma 7.3. *Let $f \in \mathscr{V}^+$ and $g \in \mathscr{V}^-$ be non-degenerate signatures of arity at least 3. Then* $\operatorname{Holant}(\{f, g\})$ *is #P-hard.*

Proof. Let $\operatorname{rd}^+(f) = d$, $\operatorname{rd}^-(g) = d'$, $\operatorname{arity}(f) = n$ and $\operatorname{arity}(g) = n'$, then $2d < n$ and $2d' < n'$. If $d \geq 2$, we can apply Lemma 4.15 zero or more times to construct a signature obtained from g by adding a certain number of self loops, and the signature is a tensor power of $[1, -i]$. To see this, note that we start with $\operatorname{rd}^-(g) < \operatorname{vd}^-(g)$ with their sum being $\operatorname{arity}(g)$. We are allowed to apply Lemma 4.15, if the signature is nonzero and its vd^- is positive. Each time we apply Lemma 4.15, we reduce rd^- and vd^- by one, and the arity by two. Thus $\mathrm{rd}^- < \mathrm{vd}^-$ is maintained until rd^- becomes 0, at which point the signature is a tensor power of $[1, -i]$. Since rd^- is positive, the signature is nonzero, thus Lemma 4.15 applies. Then by Lemma 7.1, $\operatorname{Holant}(\{f, g\})$ is #P-hard. Similarly, it is #P-hard if $d' \geq 2$. Thus we may assume that $d = d' = 1$.

We perform the $Z = \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]$ transformation

$$\begin{aligned} \operatorname{Holant}(=_2 \mid \{f, g\}) &\equiv_T \operatorname{Holant}\left([1,0,1]Z^{\otimes 2} \mid \{(Z^{-1})^{\otimes n} f, (Z^{-1})^{\otimes n'} g\}\right) \\ &\equiv_T \operatorname{Holant}\left([0,1,0] \mid \{\hat{f}, \hat{g}\}\right). \end{aligned}$$

For f with $\mathrm{rd}^+(f) = d$, we had shown that $(Z^{-1})^{\otimes n} f = \hat{f} = [\hat{f}_0, \ldots, \hat{f}_d, 0, \ldots, 0]$, where $\hat{f}_d \neq 0$. Similarly, as noted before, for g with $\mathrm{rd}^-(g) = d'$, $(Z^{-1})^{\otimes n'} g = \hat{g} = [0, \ldots, 0, \hat{g}_{d'}, \ldots, \hat{g}_0]$, where $\hat{g}_{d'} \neq 0$.

Since $d = d' = 1$, up to a nonzero constant, $\hat{f} = [a, 1, 0, \ldots, 0]$ and $\hat{g} = [0, \ldots, 0, 1, b]$, for some $a, b \in \mathbb{C}$. We show that it is always possible to get two such signatures of the same arity $\min\{n, n'\}$. Suppose $n > n'$. We form a loop from $\hat{f}$, where the loop is really a path consisting of one vertex and two edges, with the vertex assigned the signature $\neq_2$. It is easy to see that this signature is the degenerate signature $2[1,0]^{\otimes(n-2)}$. Similarly, we can form a loop from $\hat{g}$ and can get $2[0,1]^{\otimes(n'-2)}$. Thus we have both $[1,0]^{\otimes(n-2)}$ and $[0,1]^{\otimes(n'-2)}$. We can connect all $n' - 2$ edges of the second to the first, connected by $\neq_2$. This gives $[1,0]^{\otimes(n-n')}$. We can continue subtracting the smaller arity from the larger one. We continue this process in a subtractive version of the Euclidean algorithm, and end up with both $[1,0]^{\otimes t}$ and $[0,1]^{\otimes t}$, where $t = \gcd(n-2, n'-2) = \gcd(n-n', n'-2)$. In particular, $t \mid n - n'$ and by taking $(n-n')/t$ many copies of $[0,1]^{\otimes t}$, we can get $[0,1]^{\otimes(n-n')}$. Connecting this back to $\hat{f}$ via $\neq_2$, we get a symmetric signature consisting of the first n' entries of $\hat{f}$, which has the same arity as $\hat{g}$. A similar proof works when $n' > n$.

Thus without loss of generality, we may assume $n = n'$. If $a \neq 0$, then connect $[0,1]^{\otimes(n-2)}$ to $\hat{f} = [a, 1, 0, \ldots, 0]$ via $\neq_2$ we get $\hat{h} = [a, 1, 0]$. For $a \neq 0$, translating this back by Z, we have a binary signature $h \notin \mathscr{R}_2^-$ together with the given $g \in \mathscr{V}^-$. By Lemma 7.2, $\operatorname{Holant}(\{f, g\})$ is #P-hard. A similar proof works for the case $b \neq 0$.

The only case left is when $\hat{f} = [0, 1, 0, \ldots, 0]$ of arity n, and $\hat{g} = [0, \ldots, 0, 1, 0]$ of arity n. This is #P-hard by Lemma 6.12. $\square$

8 $\mathscr{A}$- and $\mathscr{P}$-transformable Signatures

In this section, we investigate the properties of $\mathscr{A}$- and $\mathscr{P}$-transformable signatures.

8.1 Characterization of $\mathscr{A}$- and $\mathscr{P}$-transformable Signatures

Recall that by definition, if a set of signatures $\mathcal{F}$ is $\mathscr{A}$-transformable (resp. $\mathscr{P}$-transformable), then the binary equality $=_2$ must be simultaneously transformed into $\mathscr{A}$ (resp. $\mathscr{P}$) along with $\mathcal{F}$. We first characterize what kind of matrices such a transformation can be by just considering the transformation of the binary equality. While there are many binary signatures in $\mathscr{A} \cup \mathscr{P}$, it turns out that it is sufficient to consider only four signatures.

Proposition 8.1. *Let $T \in \mathbb{C}^{2 \times 2}$ be a matrix and $\alpha = (1+i)/\sqrt{2} = \sqrt{i}$. Let $\mathbf{O}_2(\mathbb{C})$ denote the group of 2-by-2 orthogonal matrices over $\mathbb{C}$. Then the following hold:*

1. $[1,0,1]T^{\otimes 2}=[1,0,1]$ *iff* $T\in \mathbf{O}_2(\mathbb{C})$.
2. $[1,0,1]T^{\otimes 2}=[1,i,1]$ *iff there exist* $H\in \mathbf{O}_2(\mathbb{C})$ *such that* $T=\frac{1}{\sqrt{1-i}}H\left[\begin{smallmatrix}1 & 1\\ \alpha^3 & -\alpha^3\end{smallmatrix}\right]$.
3. $[1,0,1]T^{\otimes 2}=[0,1,0]$ *iff there exist* $H\in \mathbf{O}_2(\mathbb{C})$ *such that* $T=\frac{1}{\sqrt{2}}H\left[\begin{smallmatrix}1 & 1\\ i & -i\end{smallmatrix}\right]$.
4. $[1,0,1]T^{\otimes 2}=[1,0,\nu]$ *iff there exist* $H\in \mathbf{O}_2(\mathbb{C})$ *such that* $T=H\left[\begin{smallmatrix}1 & 0\\ 0 & \sqrt{\nu}\end{smallmatrix}\right]$.

Proof. Case 1 is clear since

$$[1,0,1]T^{\otimes 2}=[1,0,1] \iff T^{\mathrm{T}}I_2T=I_2 \iff T^{\mathrm{T}}T=I_2,$$

the definition of a (2-by-2) orthogonal matrix. Now we use this case to prove the others.

For $j\in\{2,3,4\}$, let M_j denote the matrices $\frac{1}{\sqrt{1-i}}\left[\begin{smallmatrix}1 & 1\\ \alpha^3 & -\alpha^3\end{smallmatrix}\right]$, $\frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1 & 1\\ i & -i\end{smallmatrix}\right]$, and $\left[\begin{smallmatrix}1 & 0\\ 0 & \sqrt{\nu}\end{smallmatrix}\right]$ respectively. Let $T_j=HM_j$, where H is an orthogonal matrix, then

$$[1,0,1]T_j^{\otimes 2}=[1,0,1](HM_j)^{\otimes 2}=[1,0,1]M_j^{\otimes 2}=f_j,$$

where f_j is the binary signature in case j.

On the other hand, suppose that $[1,0,1](T_j)^{\otimes 2}=f_j$. Then we have

$$[1,0,1](T_jM_j^{-1})^{\otimes 2}=f_j(M_j^{-1})^{\otimes 2}=[1,0,1],$$

so $T_jM_j^{-1}$ is an orthogonal matrix by case 1, say H. Thus $T_j=HM_j$ as desired. ☐

We also need the following lemma; the proof is direct.

Lemma 8.2. *If a symmetric signature* $f=[f_0,f_1,\ldots,f_n]$ *can be expressed in the form* $f=a[1,\lambda]^{\otimes n}+b[1,\mu]^{\otimes n}$, *for some* $a,b,\lambda,\mu\in\mathbb{C}$, *then the* f_k*'s satisfy the recurrence relation* $f_{k+2}=(\lambda+\mu)f_{k+1}-\lambda\mu f_k$ *for* $0\le k\le n-2$.

Now we can characterize the $\mathscr{A}$-transformable signatures.

Lemma 8.3. *Let* $\alpha=(1+i)/\sqrt{2}=\sqrt{i}$. *A non-degenerate symmetric signature* $f=[f_0,\ldots,f_n]$ *is* $\mathscr{A}$*-transformable iff there exists an orthogonal transformation such that after the transformation, it satisfies one of the following:*

1. *For any* $0\le k\le n-2$, $f_{k+2}=f_k$, *and*
   - $f_0=0$, *or*
   - $f_1=0$, *or*
   - $f_1=\pm i f_0\neq 0$, *or*
   - n *is odd and* $f_1=\pm(1\pm\sqrt{2})if_0\neq 0$ *(all four sign choices are permissible);*
2. *For any* $0\le k\le n-2$, $f_{k+2}=-f_k$*;*
3. *For any* $0\le k\le n-2$, $f_{k+2}=\sigma i f_k$, *where* $\sigma=\pm1$, *and*
   - $f_0=0$, *or*
   - $f_1=0$, *or*
   - $f_1=\pm\alpha i f_0\neq 0$ *(for* $\sigma=+1$*), and* $f_1=\pm\alpha f_0\neq 0$ *(for* $\sigma=-1$*).*

We call these three categories of signatures $\mathscr{A}_1$, $\mathscr{A}_2$, and $\mathscr{A}_3$ respectively.

Proof of Lemma 8.3. By definition, f is $\mathscr{A}$-transformable iff there exists a matrix T and a signature g such that $(=_2)T^{\otimes 2} \in \mathscr{A}$ and $g = (T^{-1})^{\otimes n} f \in \mathscr{A}$. Since g is symmetric, $g \in \mathscr{A}$ is equivalent to $g \in \mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$. The set of signatures $\mathscr{A}$ is closed under a scalar multiplication. Thus we list the non-degenerate symmetric binary signatures in $\mathscr{A}$ up to a scalar multiple (see Section 2.4):

$$[1, 0, \pm 1], [1, 0, \pm i], [1, \pm 1, -1], [1, \pm i, 1], [0, 1, 0]. \tag{10}$$

The set of signatures $\mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$, as column vectors, is also closed under a left multiplication by $D = \left[\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right]$. This can be seen as follows. Any signature in $\mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$ is expressible as $c(v_1^{\otimes n} + i^t v_2^{\otimes n})$, where $t \in \{0, 1, 2, 3\}$ and (v_1, v_2) is a pair of vectors in the set

$$\left\{ \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right), \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right), \left(\begin{bmatrix} 1 \\ i \end{bmatrix}, \begin{bmatrix} 1 \\ -i \end{bmatrix} \right) \right\}. \tag{11}$$

Then (Dv_1, Dv_2) is also a pair of vectors in the above set, up to a different multiplier c and i^t. Therefore $g \in \mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$ iff $D^{\otimes n} g \in \mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$. Thus we may normalize T by TD^{ℓ} ($\ell \in \{0, 1, 2, 3\}$) in consideration of (10), and only deal with those T such that

$$[1, 0, 1] T^{\otimes 2} \in \{[1, 0, 1], [1, 0, i], [1, i, 1], [0, 1, 0]\}. \tag{12}$$

Now it is clear that f is $\mathscr{A}$-transformable iff there exists a T satisfying (12) such that we can express f as $c((Tv_1)^{\otimes n} + i^t (Tv_2)^{\otimes n})$ where (v_1, v_2) is a pair of vectors in the set defined in (11).

Any matrix T satisfying (12) takes the form HN, where H is orthogonal and N is given in the four cases in Proposition 8.1. Suppose the given $T = H_0 N$, and $f = c((Tv_1)^{\otimes n} + i^t (Tv_2)^{\otimes n})$. We may further normalize it by choosing an arbitrary orthogonal H' and let $H = H' H_0^{-1}$. Then $T' = H'N$ also satisfies (12), and $H^{\otimes n} f = c((T'v_1)^{\otimes n} + i^t (T'v_2)^{\otimes n})$. We want to find an orthogonal H' such that $\hat{f} = H^{\otimes n} f$ satisfies the recurrence $\hat{f}_{k+2} = i^r \hat{f}_k$ for some $r \in \{0, 1, 2, 3\}$. By Lemma 8.2, it is sufficient to choose an orthogonal H' such that $H'Nv_1$ and $H'Nv_2$ take the form $a \left[\begin{smallmatrix} 1 \\ \lambda \end{smallmatrix}\right]$ and $b \left[\begin{smallmatrix} 1 \\ \mu \end{smallmatrix}\right]$, where $\lambda + \mu = 0$ and $\lambda\mu = i^r$ for some $r \in \{0, 1, 2, 3\}$.

Now we consider the four cases in Proposition 8.1.

1. For case 1 of Proposition 8.1, $N = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. If $[v_1\ v_2] = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, then $H' = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ gives the desired result. For the other two cases of $[v_1\ v_2]$, $H' = I_2$ gives the desired result.
2. For case 2 of Proposition 8.1, $N = \frac{1}{\sqrt{1-i}} \left[\begin{smallmatrix} 1 & 1 \\ \alpha^3 & -\alpha^3 \end{smallmatrix}\right]$. If $[v_1\ v_2] = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$, then $H' = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ gives the desired result. For the other two cases of $[v_1\ v_2]$, $H' = I_2$ gives the desired result.
3. For case 3 of Proposition 8.1, $N = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]$. If $[v_1\ v_2] = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$, then $H' = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ gives the desired result. For the other two cases of $[v_1\ v_2]$, $H' = I_2$ gives the desired result.
4. For case 4 of Proposition 8.1 with $\nu = i$, $N = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right]$. If $[v_1\ v_2] = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, then $H' = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ gives the desired result. For the other two cases of $[v_1\ v_2]$, $H' = I_2$ gives the desired result.

By examining each case separately where $\hat{f}$ has been expressed as the sum of two tensor powers, up to a global factor c, the following forms are possible:

1. $\hat{f} = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]^{\otimes n} + \beta \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]^{\otimes n}$, where $\beta = i^r$ or $i^r \alpha^n$. This is type $\mathscr{A}_1$, which has the complication when n is odd, as stated.
2. $\hat{f} = \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]^{\otimes n} + \beta \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right]^{\otimes n}$, where $\beta = i^r$. This is type $\mathscr{A}_2$. Furthermore by choosing an orthogonal H' one can make β to be any nonzero multiple of i^r. This means that any ratio between f_1 and f_0 is permissible as long as $f_1 \neq \pm i f_0$ (which together with $f_{k+2} = -f_k$ would give a degenerate signature f).

3. $\hat{f} = \left[\begin{smallmatrix}1\\ \gamma\end{smallmatrix}\right]^{\otimes n} + i^r \left[\begin{smallmatrix}1\\ -\gamma\end{smallmatrix}\right]^{\otimes n}$, where $\gamma = \alpha$ or α^3. This is type $\mathscr{A}_3$.

Conversely, if $H^{\otimes n} f$ is in one of the forms given in the lemma, for some orthogonal H, then one can directly check that f is $\mathscr{A}$-transformable. □

We also have a similar characterization for $\mathscr{P}$-transformable signatures.

Lemma 8.4. *A non-degenerate symmetric signature $f = [f_0, \ldots, f_n]$ is $\mathscr{P}$-transformable iff there exists an orthogonal transformation such that after the transformation, it satisfies one of the following:*

1. *For any $0 \le k \le n-2$, $f_{k+2} = f_k$;*
2. *For any $0 \le k \le n-2$, $f_{k+2} = -f_k$.*

We call the first category of signatures $\mathscr{P}_1$ and the second $\mathscr{P}_2$. Notice that $\mathscr{A}_1 \subset \mathscr{P}_1$ and $\mathscr{A}_2 = \mathscr{P}_2$. Also note that, since f is non-degenerate, $f_1 \neq \pm f_0$ in case 1, and $f_1 \neq \pm i f_0$ in case 2, are implied.

Proof of Lemma 8.4. By definition, f is $\mathscr{P}$-transformable iff there exists a matrix T and a signature g such that $(=_2)T^{\otimes 2} \in \mathscr{P}$ and $g = (T^{-1})^{\otimes n} f \in \mathscr{P}$. Since g is symmetric and non-degenerate, g is a generalized equality, or possibly a binary disequality if $n = 2$. We can express g as either $g = a v_1^{\otimes n} + b v_2^{\otimes n}$, where $a, b \neq 0$, $v_1 = \left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right]$, and $v_2 = \left[\begin{smallmatrix}0\\ 1\end{smallmatrix}\right]$, or when $n = 2$ there is the additional case that $g = c(v_1^{\otimes 2} - v_2^{\otimes 2})$, where $c \neq 0$, $v_1 = \left[\begin{smallmatrix}1\\ 1\end{smallmatrix}\right]$, and $v_2 = \left[\begin{smallmatrix}1\\ -1\end{smallmatrix}\right]$. This set of signatures is closed under a nonzero constant multiplier. On the other hand, $(=_2)T^{\otimes 2} \in \mathscr{P}$ means that this signature is either a generalized binary equality or a binary disequality, and thus T, after a nonzero multiplier, takes the form given in cases 3 and 4 of Proposition 8.1. To show that the recurrence $f_{k+2} = \pm f_k$ holds, by Lemma 8.2, it is sufficient to choose an orthogonal H such that HTv_1 and HTv_2 take the form, up to a nonzero multiplier, $\left[\begin{smallmatrix}1\\ \lambda\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1\\ \mu\end{smallmatrix}\right]$ respectively, where $\lambda + \mu = 0$ and $\lambda\mu = \mp 1$.

In cases 3 and 4 of Proposition 8.1, the matrix T is of the form $H_0 N$ for some orthogonal matrix H_0 and some particular matrix N. Let $H = H' H_0^{-1}$, where H' is an arbitrary 2-by-2 orthogonal matrix. Then we only need to consider $H' N v_1$ and $H' N v_2$.

1. Consider $[v_1\ v_2] = \left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]$.
   (a) In case 3 of Proposition 8.1, $N = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1 & 1\\ i & -i\end{smallmatrix}\right]$. Then $H' = I_2$ gives $H' N v_1 = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1\\ i\end{smallmatrix}\right]$ and $H' N v_2 = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1\\ -i\end{smallmatrix}\right]$, which is in $\mathscr{P}_2$.
   (b) In case 4 of Proposition 8.1, $N = \left[\begin{smallmatrix}1 & 0\\ 0 & \sqrt{\nu}\end{smallmatrix}\right]$. Then $H' = \left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]$ gives $H' N v_1 = \left[\begin{smallmatrix}1\\ 1\end{smallmatrix}\right]$ and $H' N v_2 = \sqrt{\nu}\left[\begin{smallmatrix}1\\ -1\end{smallmatrix}\right]$, which is in $\mathscr{P}_1$.
2. Consider $[v_1\ v_2] = \left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]$, which means that the arity n is 2 and $f = [0, 1, 0]$.
   (a) In case 3 of Proposition 8.1, $N = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1 & 1\\ i & -i\end{smallmatrix}\right]$. Then $H' = \left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]$ gives $H' N v_1 = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1\\ 1\end{smallmatrix}\right]$ and $H' N v_2 = \frac{i}{\sqrt{2}}\left[\begin{smallmatrix}1\\ -1\end{smallmatrix}\right]$, which is in $\mathscr{P}_1$.
   (b) In case 4 of Proposition 8.1, $N = \left[\begin{smallmatrix}1 & 0\\ 0 & \sqrt{\nu}\end{smallmatrix}\right]$. Then $H' = I_2$ gives $H' N v_1 = \left[\begin{smallmatrix}1\\ \sqrt{\nu}\end{smallmatrix}\right]$ and $H' N v_2 = \left[\begin{smallmatrix}1\\ -\sqrt{\nu}\end{smallmatrix}\right]$, so

$$g = \begin{bmatrix}1\\ \sqrt{\nu}\end{bmatrix}^{\otimes 2} - \begin{bmatrix}1\\ -\sqrt{\nu}\end{bmatrix}^{\otimes 2} = 2\sqrt{\nu}\begin{bmatrix}0\\ 1\\ 0\end{bmatrix} = \sqrt{\nu}\left(\begin{bmatrix}1\\ 1\end{bmatrix}^{\otimes 2} - \begin{bmatrix}1\\ -1\end{bmatrix}^{\otimes 2}\right),$$

which is in $\mathscr{P}_1$.

Conversely, one can directly check that the signatures listed in the lemma are $\mathscr{P}$-transformable. In fact, the transformations that we applied above are all invertible. $\square$

Combining Lemma 8.3 and Lemma 8.4, we have a necessary and sufficient condition for a single signature to be $\mathscr{A}$- or $\mathscr{P}$-transformable.

Corollary 8.5. *A signature f is $\mathscr{A}$- or $\mathscr{P}$-transformable iff $f \in \mathscr{P}_1 \cup \mathscr{P}_2 \cup \mathscr{A}_3$.*

Notice that our definitions of $\mathscr{P}_1$, $\mathscr{P}_2$, and $\mathscr{A}_3$ involve an orthogonal transformation. For any single signature $f \in \mathscr{P}_1 \cup \mathscr{P}_2 \cup \mathscr{A}_3$, Holant$(f)$ is tractable. However, this does not imply that Holant$(\mathscr{P}_1)$, Holant$(\mathscr{P}_2)$, or Holant$(\mathscr{A}_3)$ is tractable. In fact, one can check, using Theorem 5.1, that each of these problems is #P-hard.

The next two lemmas give a procedure to check if a signature is in $\mathscr{P}_1 \cup \mathscr{P}_2 \cup \mathscr{A}_3$. The first one is obvious.

Lemma 8.6. *For a symmetric signature f of arity n and a nonsingular matrix T, let $\hat{f} = T^{\otimes n} f$. Then f satisfies a second order recurrence relation iff $\hat{f}$ does as well.*

For a pair of linearly independent vectors $v_0 = [a_0, b_0]$ and $v_1 = [a_1, b_1]$, define

$$\theta(v_0, v_1) = \frac{a_0 b_1 - a_1 b_0}{a_0 a_1 + b_0 b_1},$$

which we allow to be ∞. This is well-defined; the only case this expression is not defined is when $v_0 = 0$ or $v_1 = 0$ or both v_0 and v_1 are a multiple of the same $[1, \pm i]$. Intuitively, this formula is the tangent of the angle from v_0 to v_1. (The tangent of this "complex angle" is defined in the extended Riemann complex plane $\mathbb{C} \cup \{\infty\}$.) An orthogonal transformation must keep this θ invariant or negated. Formally, we have the following lemma, which is proved by simple algebra.

Lemma 8.7. *For two linearly independent vectors v_0, $v_1 \in \mathbb{C}^2$ and an orthogonal matrix H, let $\widehat{v_0} = H v_0$ and $\widehat{v_1} = H v_1$. Then $\theta(v_0, v_1) = \pm\theta(\widehat{v_0}, \widehat{v_1})$.*

The following Proposition is easy to prove.

Proposition 8.8 (Lemma 9.11 in [16])**.** *Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ be four vectors and suppose that $\mathbf{c}, \mathbf{d}$ are linearly independent. If for some $n \geq 3$, we have $\mathbf{a}^{\otimes n} + \mathbf{b}^{\otimes n} = \mathbf{c}^{\otimes n} + \mathbf{d}^{\otimes n}$, then either $\mathbf{a} = \omega_1 \mathbf{c}$ and $\mathbf{b} = \omega_2 \mathbf{d}$ or $\mathbf{a} = \omega_1 \mathbf{d}$ and $\mathbf{b} = \omega_2 \mathbf{c}$ for some $\omega_1^n = \omega_2^n = 1$.*

Now we have some necessary conditions for a signature f to be in $\mathscr{P}_1 \cup \mathscr{P}_2 \cup \mathscr{A}_3$. Let f be a non-degenerate signature of arity $n \geq 3$. If f does not satisfy any second order recurrence relation, then by Lemma 8.6 it is not $\mathscr{A}$- or $\mathscr{P}$-transformable. Otherwise, we can express f as $v_0^{\otimes n} + v_1^{\otimes n}$, where v_0 and v_1 are linearly independent, due to f being non-degenerate. By Proposition 8.8, $\theta(v_0, v_1)$ is uniquely determined, up to a $\pm$ sign. Then by Lemma 8.7, f is $\mathscr{A}$- or $\mathscr{P}$-transformable only if $\theta(v_0, v_1)$ is one of the following values ∞, $\pm i$, or $\pm\sqrt{2}i$. We summarize this discussion as the following lemma.

Lemma 8.9. *If a non-degenerate symmetric signature f is $\mathscr{A}$- or $\mathscr{P}$-transformable, then f is of the form $v_0^{\otimes n} + v_1^{\otimes n}$ such that v_0 and v_1 are linearly independent and $\theta(v_0, v_1) \in \{\infty, \pm i, \pm\sqrt{2}i\}$.*

8.2 Dichotomies when $\mathscr{A}$- or $\mathscr{P}$-transformable Signatures Appear

Our characterization of $\mathscr{A}$-transformable and $\mathscr{A}$-transformable signatures are up to an orthogonal transformation. Since an orthogonal transformation never changes the complexity of the problem, in the following lemmas, we assume this transformation is already done.

Lemma 8.10. *Let $\mathcal{F}$ be a set of symmetric signatures. Suppose $\mathcal{F}$ contains a non-degenerate signature $f \in \mathscr{P}_1$ of arity $n \geq 3$. Then* $\operatorname{Holant}(\mathcal{F})$ *is* #P*-hard unless $\mathcal{F}$ is $\mathscr{P}$-transformable or $\mathscr{A}$-transformable.*

Proof. By assumption, for any $0 \leq k \leq n-2$, $f_{k+2} = f_k$ and $f_1 \neq \pm f_0$ since f is not degenerate. We can express f as

$$f = a_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix}^{\otimes n} + a_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}^{\otimes n},$$

where $a_0 = (f_0 + f_1)/2$ and $a_1 = (f_0 - f_1)/2$. For this f, we can further perform an orthogonal transformation by $H = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ so that f is transformed into the generalized equality signature $2^n[a_0, 0, \ldots, 0, a_1]$ of arity n, where $a_0 a_1 \neq 0$. By Lemma A.2, we can obtain $=_4$, the arity 4 equality signature.

Since we can always realize the arity 4 equality, we can realize any equality signature of even arity. Thus, $\#\mathrm{CSP}^2(\mathcal{F}) \leq_T \operatorname{Holant}(\mathcal{F})$. By Theorem 2.10, the $\#\mathrm{CSP}^d$ dichotomy, $\operatorname{Holant}(\mathcal{F})$ is #P-hard unless $T\mathcal{F}$ is in $\mathscr{A}$ or $\mathscr{P}$ where T is of the form $\left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha^k \end{smallmatrix}\right]$ for an integer $0 \leq k \leq 7$.

If $T\mathcal{F} \subseteq \mathscr{P}$, then we have $\mathcal{F} \subseteq T^{-1}\mathscr{P}$. Notice that $T^{-1}\mathscr{P} = \mathscr{P}$. So the original $\mathcal{F}$ is $\mathscr{P}$-transformable after some orthogonal transformation. Otherwise, $T\mathcal{F} \subseteq \mathscr{A}$. It is easy to verify that $(=_2)(T^{-1})^{\otimes 2}$ is $[1, 0, i^{8-k}] \in \mathscr{A}$. Thus $\mathcal{F}$ is $\mathscr{A}$-transformable under some orthogonal transformation. □

Lemma 8.11. *Let $\mathcal{F}$ be a set of symmetric signatures. Suppose $\mathcal{F}$ contains a non-degenerate signature $f \in \mathscr{P}_2$ of arity $n \geq 3$. Then* $\operatorname{Holant}(\mathcal{F})$ *is* #P*-hard unless $\mathcal{F}$ is $\mathscr{P}$-transformable or $\mathscr{A}$-transformable.*

Proof. By assumption, for any $0 \leq k \leq n-2$, $f_{k+2} = -f_k$ and $f_1 \neq \pm i f_0$ since f is not degenerate. We can express f as

$$f = a_0 \begin{bmatrix} 1 \\ i \end{bmatrix}^{\otimes n} + a_1 \begin{bmatrix} 1 \\ -i \end{bmatrix}^{\otimes n},$$

where $a_0 = (f_0 + i f_1)/2$ and $a_1 = (f_0 - i f_1)/2$, and $a_0, a_1 \neq 0$. Then under the holographic transformation $Z = \left[\begin{smallmatrix} \sqrt[n]{a_0} & \sqrt[n]{a_1} \\ \sqrt[n]{a_0} i & -\sqrt[n]{a_1} i \end{smallmatrix}\right]^{-1}$, we have

$$Z^{\otimes n} f = (=_n) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\otimes n} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}^{\otimes n}$$

and

$$\begin{aligned} \operatorname{Holant}(=_2 \mid \mathcal{F} \cup \{f\}) &\equiv_T \operatorname{Holant}\left([1,0,1](Z^{-1})^{\otimes 2} \mid Z\mathcal{F} \cup \{Z^{\otimes n} f\}\right) \\ &\equiv_T \operatorname{Holant}\left([0,1,0] \mid Z\mathcal{F} \cup \{=_n\}\right). \end{aligned}$$

Thus, we have a bipartite graph with $=_n$ on the right and $(\neq_2) = [0,1,0]$ on the left, so all equalities of arity a multiple of n are realizable on the right side. Moreover, we can move these equalities to the left side since the binary disequality just reverses these signatures (exchanging input bits 0's and 1's), which leaves the equalities unchanged.

Now we can apply Theorem 2.10, the $\#\mathrm{CSP}^d$ dichotomy. Let ω be a primitive $4n$-th root of unity. Then under the holographic transformation $T = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \omega^k \end{smallmatrix}\right]$ for some integer k, $TZ\mathcal{F}$ must be in $\mathscr{A}$ or $\mathscr{P}$. However, if $TZ\mathcal{F} \subseteq \mathscr{P}$, then we have $Z\mathcal{F} \subseteq T^{-1}\mathscr{P}$. Notice that $T^{-1}\mathscr{P} = \mathscr{P}$. So the $\mathcal{F}$ is $\mathscr{P}$-transformable under this Z transformation.

Otherwise, $TZ\mathcal{F} \subseteq \mathscr{A}$. It is easy to verify that $(=_2)((TZ)^{-1})^{\otimes 2}$ is still $[0,1,0] \in \mathscr{A}$. Thus $\mathcal{F}$ is $\mathscr{A}$-transformable under this TZ transformation. ☐

Lemma 8.12. *Let $\mathcal{F}$ be a set of symmetric signatures. Suppose $\mathcal{F}$ contains a non-degenerate signature $f \in \mathscr{A}_3$ of arity $n \geq 3$. Then* Holant$(\mathcal{F})$ *is* #P*-hard unless $\mathcal{F}$ is $\mathscr{A}$-transformable.*

Proof. By assumption, for any $0 \leq k \leq n-2$, we have $f_{k+2} = \pm i f_k$. We consider $f_{k+2} = i f_k$ since the other case is similar. We can express f as

$$f = a_0 \begin{bmatrix} 1 \\ \alpha \end{bmatrix}^{\otimes n} + a_1 \begin{bmatrix} 1 \\ -\alpha \end{bmatrix}^{\otimes n},$$

where $a_1/a_0 = i^r$ for some integer r.

A self loop on f yields f', where $f'_k = f_k + f_{k+2} = (1+i)f_k$. Thus up to the constant $(1+i)$, f' is just the first $n-2$ entries of f. By doing more self loops, we eventually obtain an arity 4 signature when n is even or a ternary one when n is odd. There are eight cases depending on the first two entries of f and the parity of n. However, for any case, we can realize the signature $[1,0,i]$. We list them here. (In the calculations below, we cancel certain nonzero constant factors without explanation.)

- $[0,1,0,i]$: Another self loop gives $[0,1]$. Connect it back to the ternary and we get $[1,0,i]$.
- $[1,0,i,0]$: Another self loop gives $[1,0]$. Connect it back to the ternary and we get $[1,0,i]$.
- $[1,\alpha i,i,-\alpha]$: Another self loop gives $[1,\alpha i]$. Connect two copies of it to the ternary and we get $[1,-\alpha]$. Then connect this back to the ternary to finally get $[1,0,i]$. See Figure 12a.
- $[1,-\alpha i,i,\alpha]$: Same construction as the previous case.
- $[0,1,0,i,0]$: Another self loop gives $[0,1,0]$. Connect it back to this arity 4 signature and get $[1,0,i]$.
- $[1,0,i,0,-1]$: Another self loop gives $[1,0,i]$ directly.
- $[1,\alpha i,i,-\alpha,-1]$: Another self loop gives $[1,\alpha i,i]$. Connect two copies of it together to get $[1,-\alpha,-i]$. Connect this to the arity 4 signature to get $[1,0,i]$. See Figure 12b.
- $[1,-\alpha i,i,\alpha,-1]$: Same construction as the previous case.

(a) Vertices assigned $[1,\alpha i,i,-\alpha]$. (b) Vertices assigned $[1,\alpha i,i,-\alpha,-1]$.

Figure 12: Constructions to realize $[1,0,i]$.

With $[1,0,i]$ in hand, we can connect three copies to get $[1,0,-i]$.[1] Now we construct a bipartite graph, with $\mathcal{F}\cup\{=_2\}$ on the right side and $[1,0,-i]$ on the left, and do a holographic transformation by $Z = \left[\begin{smallmatrix} \alpha & 1 \\ -\alpha & 1 \end{smallmatrix}\right]$ to get

$$\begin{aligned} \operatorname{Holant}([1,0,-i] \mid \mathcal{F}\cup\{f,=_2\}) &\equiv_T \operatorname{Holant}\left([1,0,-i](Z^{-1})^{\otimes 2} \mid Z\mathcal{F}\cup\{Z^{\otimes n}f, Z^{\otimes 2}(=_2)\}\right) \\ &\equiv_T \operatorname{Holant}\left(\frac{1}{2i}[1,0,1] \mid Z\mathcal{F}\cup\{[1,0,\ldots,0,i^k],[1,-i,1]\}\right) \\ &\equiv_T \operatorname{Holant}\left(Z\mathcal{F}\cup\{[1,0,\ldots,0,i^k],[1,-i,1]\}\right). \end{aligned}$$

Notice that f becomes $[1,0,\ldots,0,i^k]$ where $k = r+2n$ (after normalizing the first entry) and $=_2$ becomes $[1,-i,1]$. On the other side, $[1,0,-i]$ becomes $[1,0,1]$. Therefore, we can construct all equalities of even arity using the powers of the transformed f. Now by the #CSP^d dichotomy, Theorem 2.10, if $\mathcal{F}$ is not hard, $Z\mathcal{F}\cup\{[1,-i,1]\}$ must be #CSP^2 tractable. Therefore there exists some T of the form $\left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha^d \end{smallmatrix}\right]$, where the integer $d \in \{0,1,\ldots,7\}$, such that $TZ\mathcal{F}\cup\{T^{\otimes 2}[1,-i,1]\}$ is contained in $\mathscr{A}$ or $\mathscr{P}$.

However, $T^{\otimes 2}[1,-i,1]$ can never be in $\mathscr{P}$. Thus $TZ\mathcal{F}\cup\{T^{\otimes 2}[1,-i,1]\}\subset\mathscr{A}$. Further notice that if $d\in\{1,3,5,7\}$ in the expression of T, then $T^{\otimes 2}[1,-i,1]$ is not in $\mathscr{A}$. Hence T must be of the form $\left[\begin{smallmatrix} 1 & 0 \\ 0 & i^d \end{smallmatrix}\right]$ where the integer $d\in\{0,1,2,3\}$. For such T, $T^{\otimes 2}[1,-i,1]\in\mathscr{A}$, and $T^{-1}\mathscr{A}=\mathscr{A}$. Thus $TZ\mathcal{F}\cup\{T^{\otimes 2}[1,-i,1]\}\subset\mathscr{A}$ becomes just $Z\mathcal{F}\subset\mathscr{A}$. Moreover, $(=_2)(Z^{-1})^{\otimes 2}$ is $[1,i,1]\in\mathscr{A}$. Thus $\mathcal{F}$ is $\mathscr{A}$-transformable under this Z transformation. □

9 The Main Dichotomy

In this section, we prove our main dichotomy theorem. We begin with a dichotomy for a single signature, which we prove by induction on its arity.

Theorem 9.1. *If f is a non-degenerate symmetric signature of arity at least 3 with complex weights in Boolean variables, then* Holant(f) *is* #P*-hard unless $f\in\mathscr{P}_1\cup\mathscr{P}_2\cup\mathscr{A}_3$ or f is vanishing, in which case the problem is in* P*.*

Recall that $\mathscr{A}_1\subset\mathscr{P}_1$ and $\mathscr{A}_2=\mathscr{P}_2$. Thus $f\in\mathscr{P}_1\cup\mathscr{P}_2\cup\mathscr{A}_3$ iff f is $\mathscr{A}$-transformable or $\mathscr{P}$-transformable by Lemmas 8.3 and 8.4.

Proof. Let the arity of f be n. The base cases of $n=3$ and $n=4$ are proved in Theorems 2.8 and 6.11 respectively. Now assume $n\geq 5$.

With the signature f, we form a self loop to get a signature f' of arity at least 3. We consider the cases separately whether f' is degenerate or not.

- Suppose $f'=[a,b]^{\otimes(n-2)}$ is degenerate. There are three cases to consider.
    1. If $a=b=0$, then f' is the all zero signature. For f, this means $f_{k+2}=-f_k$ for $0\leq k\leq n-2$, so $f\in\mathscr{P}_2$ by Lemma 8.4, and therefore Holant(f) is tractable.
    2. If $a^2+b^2\neq 0$, then f' is nonzero and $[a,b]$ is not a constant multiple of either $[1,i]$ or $[1,-i]$. We may normalize so that $a^2+b^2=1$. Then the orthogonal transformation

[1] In the other case of $f_{k+2}=-if_k$, we get $[1,0,-i]$ directly here.

$\left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right]$ transforms the column vector $[a, b]$ to $[1, 0]$. Let $\hat{f}$ be the transformed signature from f, and $\widehat{f'} = [1, 0]^{\otimes(n-2)}$ the transformed signature from f'.

Since an orthogonal transformation keeps $=_2$ invariant, this transformation commutes with the operation of taking a self loop, i.e., $\widehat{f'} = (\hat{f})'$. Here $(\hat{f})'$ is the function obtained from $\hat{f}$ by taking a self loop. So $\hat{f}_0 + \hat{f}_2 = 1$ and for every integer $1 \le k \le n-2$, we have $\hat{f}_k = -\hat{f}_{k+2}$. With one or more self loops, we eventually obtain $[1, 0]$ or $[1, 0, 0]$ depending on the parity of n. In either case, we connect an appropriate number of copies of this signature to $\hat{f}$ to get a arity 4 signature $\hat{g} = [\hat{f}_0, \hat{f}_1, \hat{f}_2, -\hat{f}_1, -\hat{f}_2]$. We show that $\operatorname{Holant}(\hat{g})$ is #P-hard. To see this, we first compute $\det(\widetilde{M_g}) = -2(\hat{f}_0 + \hat{f}_2)(\hat{f}_1^2 + \hat{f}_2^2) = -2(\hat{f}_1^2 + \hat{f}_2^2)$, since $\hat{f}_0 + \hat{f}_2 = 1$. Therefore if $\hat{f}_1^2 + \hat{f}_2^2 \neq 0$, $\operatorname{Holant}(\hat{g})$ is #P-hard by Corollary 6.7. Otherwise $\hat{f}_1^2 + \hat{f}_2^2 = 0$, and we consider $\hat{f}_2 = i\hat{f}_1$ since the other case is similar. Since f is non-degenerate, $\hat{f}$ is non-degenerate, which implies $\hat{f}_2 \neq 0$. We can express $\hat{g}$ as $[1, 0]^{\otimes 4} - \hat{f}_2[1, i]^{\otimes 4}$. Under the holographic transformation by $T = \left[\begin{smallmatrix} 1 & (-\hat{f}_2)^{1/4} \\ 0 & i(-\hat{f}_2)^{1/4} \end{smallmatrix}\right]$, we have

$$\begin{aligned} \operatorname{Holant}\left(=_2 \mid \hat{g}\right) &\equiv_T \operatorname{Holant}\left([1, 0, 1]T^{\otimes 2} \mid (T^{-1})^{\otimes 4}\hat{g}\right) \\ &\equiv_T \operatorname{Holant}\left(\hat{h} \mid =_4\right), \end{aligned}$$

where

$$\hat{h} = [1, 0, 1]T^{\otimes 2} = [1, (-\hat{f}_2)^{1/4}, 0]$$

and $\hat{g}$ is transformed by T^{-1} into the arity 4 equality $=_4$, since

$$T^{\otimes 4}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\otimes 4} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}^{\otimes 4}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\otimes 4} - \hat{f}_2 \begin{bmatrix} 1 \\ i \end{bmatrix}^{\otimes 4} = \hat{g}.$$

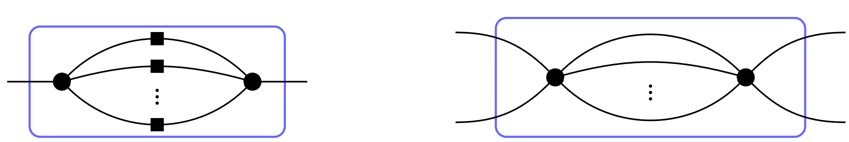
By Theorem 2.9′, $\operatorname{Holant}\left(\hat{h} \mid =_4\right)$ is #P-hard as $\hat{f}_2 \neq 0$.

3. If $a^2 + b^2 = 0$ but $(a, b) \neq (0, 0)$, then $[a, b]$ is a nonzero multiple of $[1, \pm i]$. Ignoring the constant multiple, we have $f' = [1, i]^{\otimes(n-2)}$ or $[1, -i]^{\otimes(n-2)}$. We consider the first case since the other case is similar.

   In the first case, the characteristic polynomial of the recurrence relation of f' is $x - i$, so that of f is $(x - i)(x^2 + 1) = (x - i)^2(x + i)$. Hence there exist a_0, a_1 and c such that

$$f_k = (a_0 + a_1 k)i^k + c(-i)^k$$

   for every integer $0 \le k \le n$. If $a_1 = 0$, then f' is the all zero signature, a contradiction. If $c = 0$, then f is vanishing, one of the tractable cases. Now we assume that $a_1 c \neq 0$



(a) The circles are assigned $\hat{f}$ and the squares are assigned $\neq_2$.

(b) The circles are assigned f.

Figure 13: Two gadgets used when $f' = [1, \pm i]^{\otimes(n-2)}$.

and show that f is #P-hard. Under the holographic transformation $Z = \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]$, we have

$$\begin{aligned} \text{Holant}\,(=_2 \mid f) &\equiv_T \text{Holant}\left([1,0,1]Z^{\otimes 2} \mid (Z^{-1})^{\otimes n} f\right) \\ &\equiv_T \text{Holant}\left(2[0,1,0] \mid \hat{f}\right), \end{aligned}$$

where $\hat{f}$ takes the form $[\hat{f}_0, \hat{f}_1, 0, \ldots, 0, c]$ with $\hat{f}_1 \neq 0$, since $\hat{f}$ is the Z^{-1}-transformation of the sum of two signatures, with $\text{rd}^+ = 1$ and $\text{rd}^- = 0$ respectively. On the other side, $(=_2) = [1,0,1]$ is transformed into $(\neq_2) = [0,1,0]$ after ignoring the constant factor 2. Now consider the gadget in Figure 13a with $\hat{f}$ assigned to both vertices. This gadget has the binary signature $[0, c\hat{f}_0, 2c\hat{f}_1]$, which is equivalent to $[0, \hat{f}_0, 2\hat{f}_1]$ since $c \neq 0$. Translating back by Z to the original setting, this signature is $g = [\hat{f}_0 + \hat{f}_1, -i\hat{f}_1, \hat{f}_0 - \hat{f}_1]$. This can be verified as

$$\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} 0 & \hat{f}_0 \\ \hat{f}_0 & 2\hat{f}_1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}^{\mathtt{T}} = 2 \begin{bmatrix} \hat{f}_0 + \hat{f}_1 & -i\hat{f}_0 \\ -i\hat{f}_0 & \hat{f}_0 - \hat{f}_1 \end{bmatrix}.$$

Since $\hat{f}_1 \neq 0$, it can be directly checked that $g \notin \mathscr{R}_2^+$.

If $\hat{f}_0 \neq 0$, then g is non-degenerate. By Lemma 7.2, $\text{Holant}(\{f', g\})$ is #P-hard, hence $\text{Holant}(f)$ is also #P-hard.

Suppose $\hat{f}_0 = 0$. Then we have $g = [1, -i]^{\otimes 2}$ after ignoring the nonzero factor $\hat{f}_1$. Connecting this degenerate signature to f, we get a signature $h = \langle f, g \rangle$. We note that g annihilates the signature $c[1,-i]^{\otimes n}$, and thus $h = \langle f^*, g \rangle$, where f^* is the first summand of f, i.e., $f_k^* = (a_0 + a_1 k) i^k$ $(0 \leq k \leq n)$. Then $\text{rd}^+(f^*) = 1$, $\text{vd}^+(g) = 0$, and we can apply Lemma 4.14. It follows that $\text{rd}^+(h) = 1$ and $\text{arity}(h) \geq 3$. This implies that h is non-degenerate and $h \in \mathscr{V}^+$.

Moreover, assigning f to both vertices in the gadget of Figure 13b, we get a non-degenerate signature $h' \in \mathscr{V}^-$ of arity 4. To see this, consider this gadget after a holographic transformation by Z. In this bipartite setting, it is the same as assigning $\hat{f} = [0, \hat{f}_1, 0, \ldots, 0, c]$ (or equivalently $[0, 1, 0, \ldots, 0, c']$, where $c' = c/\hat{f}_1 \neq 0$) to both the circle and triangle vertices in the gadget of Figure 7a. The square vertices there are still assigned $(\neq_2) = [0,1,0]$. While it is not apparent from the gadget's geometry, this signature is in fact symmetric. In particular, its values on inputs 1010 and 1100 are both zero. The resulting signature is $\widehat{h'} = (Z^{-1})^{\otimes 4} h' = [0,0,0,c',0]$. Hence $\text{rd}^-(h') = 1$, and therefore h' is non-degenerate and $h' \in \mathscr{V}^-$.

By Lemma 7.3, $\text{Holant}(\{h, h'\})$ is #P-hard, hence $\text{Holant}(f)$ is also #P-hard.

- Suppose f' is non-degenerate. If f' is not in one of the tractable cases, then $\text{Holant}(f')$ is #P-hard and so is $\text{Holant}(f)$. We now assume $\text{Holant}(f')$ is not #P-hard. Then, by inductive hypothesis, $f' \in \mathscr{P}_1 \cup \mathscr{P}_2 \cup \mathscr{A}_3$ or f' is vanishing. If $f' \in \mathscr{P}_1 \cup \mathscr{P}_2 \cup \mathscr{A}_3$, then applying Lemma 8.10, Lemma 8.11, or Lemma 8.12 to f' and the set $\{f, f'\}$, we have that f is $\mathscr{A}$- or $\mathscr{P}$-transformable, so by Corollary 8.5, $f \in \mathscr{P}_1 \cup \mathscr{P}_2 \cup \mathscr{A}_3$.

  Otherwise, f' is vanishing, so $f' \in \mathscr{V}^\sigma$ for $\sigma \in \{+,-\}$ by Theorem 4.13. For simplicity, assume that $f' \in \mathscr{V}^+$. The other case is similar. Let $\text{rd}^+(f') = d - 1$, where $2d < n$ and $d \geq 2$ since f' is non-degenerate. Then the entries of f' can be expressed as

$$f'_k = i^k q(k),$$

where $q(x)$ is a polynomial of degree exactly $d-1$. However, notice that if f' satisfies some recurrence relation with characteristic polynomial $t(x)$, then f satisfies a recurrence relation with characteristic polynomial $(x^2+1)t(x)$. In this case, $t(x) = (x-i)^d$. Then the corresponding characteristic polynomial of f is $(x-i)^{d+1}(x+i)$, and thus the entries of f are

$$f_k = i^k p(k) + c(-i)^k$$

for some constant c and a polynomial $p(x)$ of degree at most d. However, the degree of $p(x)$ is exactly d, otherwise the polynomial $q(x)$ for f' would have degree less than $d-1$. If $c = 0$, then f is vanishing, a tractable case. Now assume $c \neq 0$, and we want to show the problem is #P-hard.

Thus, under the transformation $Z = \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]$, we have

$$\begin{aligned} \operatorname{Holant}(=_2 \mid f) &\equiv_T \operatorname{Holant}\left([1,0,1]Z^{\otimes 2} \mid (Z^{-1})^{\otimes n} f\right) \\ &\equiv_T \operatorname{Holant}\left(2[0,1,0] \mid \hat{f}\right), \end{aligned}$$

where $\hat{f} = [\hat{f}_0, \hat{f}_1, \dots, \hat{f}_d, 0, \dots, 0, c]$, with $\hat{f}_d \neq 0$. Taking a self loop in the original setting is equivalent to connecting $[0,1,0]$ to a signature after this transformation. Thus, doing this once on $\hat{f}$, we can get $\widehat{f'} = [\hat{f}_1, \dots, \hat{f}_d, 0, \dots, 0]$ corresponding to f', and doing this $d-2$ times on $\hat{f}$, we get a signature $\hat{h} = [\hat{f}_{d-2}, \hat{f}_{d-1}, \hat{f}_d, 0, \dots, 0, 0/c]$ of arity $n-2(d-2) = n-2d+4$. The last entry is c when $d = 2$ and is 0 when $d > 2$.

As $n > 2d$, we may do two more self loops and get $[\hat{f}_d, 0, \dots, 0]$ of arity $k = n-2d$. Now connect this signature back to $\hat{f}$ via $[0,1,0]$. It is the same as getting the last $n-k = 2d$ signature entries of $\hat{f}$. We may repeat this operation zero or more times until the arity k' of the resulting signature is less than or equal to k. We claim that this signature has the form $\hat{g} = [0, \dots, 0, c]$. In other words, the $k'+1$ entries of $\hat{g}$ consist of the last c and k' many 0's in the signature $\hat{f}$, all appearing after $\hat{f}_d$. This is because there are $n-1-d$ many 0 entries in the signature $\hat{f}$ after $\hat{f}_d$, and $n-1-d \geq k \geq k'$.

Translating back by the Z transformation, having both $[\hat{f}_d, 0, \dots, 0]$ of arity k and $\hat{g} = [0, \dots, 0, c]$ of arity k' is equivalent to, in the original setting, having both $[1, i]^k$ and $[1, -i]^{k'}$. If $k > k'$, then we can connect $[1,-i]^{k'}$ to $[1,i]^k$ and get $[1,i]^{k-k'}$. Replacing k by $k-k'$ we can repeat this process until the new $k \leq k'$. If the new $k < k'$ we can continue as in the subtractive Euclid algorithm. Keep doing this procedure and eventually we get $[1,i]^t$ and $[1,-i]^t$, where $t = \gcd(k, k')$, where $k = n-2d$ and $k' \leq k$, as defined in the previous paragraph. Now putting k/t many copies of $[1,-i]^t$ together, we get $[1,-i]^k$.

In the transformed setting, $[1,-i]^k$ is $[0, \dots, 0, 1]$ of arity k. Then we connect this back to $\hat{h}$ via $[0,1,0]$. Doing this is the same as forcing k connected edges of h be assigned 0, because $[0,1,0]$ flips the assigned value 1 in $[0, \dots, 0, 1]$ to 0. Thus we get a signature of arity $n-2d+4-k = 4$, which is $[\hat{f}_{d-2}, \hat{f}_{d-1}, \hat{f}_d, 0, 0]$. Note that the last entry is 0 (and not c), because $k \geq 1$.

However, $\operatorname{Holant}\left([0,1,0] \mid [\hat{f}_{d-2}, \hat{f}_{d-1}, \hat{f}_d, 0, 0]\right)$ is equivalent to $\operatorname{Holant}([0,1,0] \mid [0,0,1,0,0])$ when $\hat{f}_d \neq 0$, which is transformed back by Z to $\operatorname{Holant}([3,0,1,0,3])$. This is the Eulerian Orientation problem on 4-regular graphs and is #P-hard by Theorem 6.5. $\square$

Now we are ready to finish the proof of our main theorem.

Proof of hardness for Theorem 5.1. Assume that $\operatorname{Holant}(\mathcal{F})$ is not #P-hard. If all of the non-degenerate signatures in $\mathcal{F}$ are of arity at most 2, then the problem is tractable case 1. Otherwise we have some non-degenerate signatures of arity at least 3. For any such f, by Theorem 9.1, $f \in \mathscr{P}_1 \cup \mathscr{P}_2 \cup \mathscr{A}_3$ or f is vanishing. If any of them is in $\mathscr{P}_1 \cup \mathscr{P}_2 \cup \mathscr{A}_3$, then by Lemma 8.10, Lemma 8.11, or Lemma 8.12, we have that $\mathcal{F}$ is $\mathscr{A}$- or $\mathscr{P}$-transformable, which are tractable cases 2 and 3.

Now we assume all non-degenerate signatures of arity at least 3 in $\mathcal{F}$ are vanishing, and there is a nonempty set of such signatures in $\mathcal{F}$. By Lemma 7.3, they must all be in $\mathscr{V}^\sigma$ for the same $\sigma \in \{+,-\}$. By Lemma 7.2, we know that any non-degenerate binary signature in $\mathcal{F}$ has to be in $\mathscr{R}_2^\sigma$. Furthermore, if there is an $f \in \mathscr{V}^\sigma$ from $\mathcal{F}$ such that $\operatorname{rd}^\sigma(f) \geq 2$, then by Lemma 7.1, the only unary signature that is allowed in $\mathcal{F}$ is $[1, \sigma i]$, and all degenerate signatures in $\mathcal{F}$ are a tensor product of $[1, \sigma i]$. Thus, all non-degenerate signatures of arity at least 3 as well as all degenerate signatures belong to $\mathscr{V}^\sigma$, and all non-degenerate binary signatures belong to $\mathscr{R}_2^\sigma$. This is tractable case 4.

Finally, we have the following: (i) all non-degenerate signatures of arity at least 3 in $\mathcal{F}$ belong to $\mathscr{V}^\sigma$; (ii) all signatures $f \in \mathcal{F} \cap \mathscr{V}^\sigma$ have $\operatorname{rd}^\sigma(f) \leq 1$, which implies that $f \in \mathscr{R}_2^\sigma$; and (iii) all non-degenerate binary signatures in $\mathcal{F}$ belong to $\mathscr{R}_2^\sigma$. Hence all non-degenerate signatures in $\mathcal{F}$ belong to $\mathscr{R}_2^\sigma$. All unary signatures also belong to $\mathscr{R}_2^\sigma$ by definition. This is indeed tractable case 5. The proof is complete. □

From the proof of our main theorem, Theorem 5.1, the tractability criterion is decidable in polynomial time in the size of the given signature set $\mathcal{F}$.

Theorem 9.2. *Given any finite set $\mathcal{F}$ of symmetric, complex-valued signatures in Boolean variables, it is decidable in polynomial time in the size of $\mathcal{F}$ whether it satisfies the dichotomy criterion in Theorem 5.1 for* $\operatorname{Holant}(\mathcal{F})$.

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A Simple Interpolations

In addition to the two arity 4 interpolations in Section 6, we also use interpolation in the proofs of two other lemmas. Compared to our arity 4 interpolations, these binary interpolations are much simpler.

Lemma A.1. *Let $x \in \mathbb{C}$. If $x \neq 0$, then for any set $\mathcal{F}$ containing $[x, 1, 0]$, we have*

$$\operatorname{Holant}(\neq_2 \mid \mathcal{F} \cup \{[v, 1, 0]\}) \leq_T \operatorname{Holant}(\neq_2 \mid \mathcal{F})$$

for any $v \in \mathbb{C}$.

Proof. Consider an instance Ω of $\operatorname{Holant}(\neq_2 \mid \mathcal{F} \cup \{[v, 1, 0]\})$. Suppose that $[v, 1, 0]$ appears n times in Ω. We stratify the assignments in Ω based on the assignments to $[v, 1, 0]$. We only need to consider the Hamming weight zero and Hamming weight one assignments since a Hamming two assignment contributes a factor of 0. Let i be the number of Hamming weight zero assignments to $[v, 1, 0]$ in Ω. Then there are $n - i$ Hamming weight one assignments and the Holant on Ω is

$$\operatorname{Holant}_\Omega = \sum_{i=0}^{n} v^i c_i,$$

where c_i is the sum over all such assignments of the product of evaluations of all other signatures on Ω.

We construct from Ω a sequence of instances Ω_s of $\operatorname{Holant}(\mathcal{F})$ indexed by $s \geq 1$. We obtain Ω_s from Ω by replacing each occurrence of $[v, 1, 0]$ with a gadget g_s created from s copies of $[x, 1, 0]$, connected sequentially but with $(\neq_2) = [0, 1, 0]$ between each sequential pair. The signature of g_s is $[sx, 1, 0]$, which can be verified by the matrix product

$$\left(\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)^{s-1} \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}^{s-1} \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & (s-1)x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} sx & 1 \\ 1 & 0 \end{bmatrix}.$$

The Holant on Ω_s is

$$\operatorname{Holant}_{\Omega_s} = \sum_{i=0}^{n} (sx)^i c_i.$$

For $s \geq 1$, this gives a coefficient matrix that is Vandermonde. Since x is nonzero, sx is distinct for each s. Therefore, the Vandermonde system has full rank. We can solve for the unknowns c_i and obtain the value of $\operatorname{Holant}_\Omega$. □

Lemma A.2. *Let $a, b \in \mathbb{C}$. If $ab \neq 0$, then for any set $\mathcal{F}$ of complex-weighted signatures containing $[a, 0, \ldots, 0, b]$ of arity $r \geq 3$,*

$$\operatorname{Holant}(\mathcal{F} \cup \{=_4\}) \leq_T \operatorname{Holant}(\mathcal{F}).$$

Proof. Since $a \neq 0$, we can normalize the first entry to get $[1, 0, \ldots, 0, x]$, where $x \neq 0$. First, we show how to obtain an arity 4 generalized equality signature. If $r = 3$, then we connect two copies together by a single edge to get an arity 4 signature. For larger arities, we form self-loops until realizing a signature of arity 3 or 4. By this process, we have a signature $g = [1, 0, 0, 0, y]$, where $y \neq 0$. If y is a pth root of unity, then we can directly realize $=_4$ by connecting p copies of g together, two edges at a time as in Figure 3. Otherwise, y is not a root of unity and we can interpolate $=_4$ as follows.

Consider an instance Ω of $\operatorname{Holant}(\mathcal{F} \cup \{=_4\})$. Suppose that $=_4$ appears n times in Ω. We stratify the assignments in Ω based on the assignments to $=_4$. We only need to consider the all-zero and all-one assignments since any other assignment contributes a factor of 0. Let i be the number of all-one assignments to $=_4$ in Ω. Then there are $n - i$ all-zero assignments and the Holant on Ω is

$$\operatorname{Holant}_\Omega = \sum_{i=0}^{n} c_i,$$

where c_i is the sum over all such assignments of the product of evaluations of all other signatures on Ω.

We construct from Ω a sequence of instances Ω_s of $\operatorname{Holant}(\mathcal{F})$ indexed by $s \geq 1$. We obtain Ω_s from Ω by replacing each occurrence of $=_4$ with a gadget g_s created from s copies of $[1, 0, 0, 0, y]$, connecting two edges together at a time as in Figure 3. The Holant on Ω_s is

$$\operatorname{Holant}_{\Omega_s} = \sum_{i=0}^{n} (y^s)^i c_i.$$

For $s \geq 1$, this gives a coefficient matrix that is Vandermonde. Since y is neither zero nor a root of unity, y^s is distinct for each s. Therefore, the Vandermonde system has full rank. We can solve for the unknowns c_i and obtain the value of $\operatorname{Holant}_\Omega$. □

Notice that the gadget constructions are planar, so this lemma also holds when restricted to planar graphs.

B An Orthogonal Transformation

In this section, we give the details of the *orthogonal* transformation used in the proof of Lemma 6.10. We state the general case for symmetric signatures of arity n. The special case of $n = 3$ was proved in Appendix D in [8].

We are given a symmetric signature $f = [f_0, \ldots, f_n]$ such that $f_k = ck\alpha^{k-1} + d\alpha^k$, where $c \neq 0$, and $\alpha \neq \pm i$. Let $S = \left[\begin{smallmatrix} 1 & \frac{d-1}{n} \\ \alpha & c + \frac{d-1}{n}\alpha \end{smallmatrix}\right]$. Note that $\det S = c \neq 0$. Then the signature f can be expressed as

$$f = S^{\otimes n}[1, 1, 0, \ldots, 0],$$

where $[1,1,0,\ldots,0]$ should be understood as a column vector of dimension 2^n, which has a 1 in entries with index weight at most one and 0 elsewhere. This identity can be verified by observing that

$$[1,1,0,\ldots,0] = [1,0]^{\otimes n} + \frac{1}{(n-1)!}\operatorname{Sym}_n^{n-1}([1,0];[0,1])$$

and we apply $S^{\otimes n}$ using properties of tensor product, $S^{\otimes n}[1,0]^{\otimes n} = (S[1,0])^{\otimes n}$, etc. We consider the value at index $0^{n-k}1^k$, which is the same as the value at any entry of weight k. By considering where the tensor product factor $[0,1]$ is located among the n possible locations, we get

$$\alpha^k + k\left(c + \frac{d-1}{n}\alpha\right)\alpha^{k-1} + (n-k)\frac{d-1}{n}\alpha^k = ck\alpha^{k-1} + d\alpha^k.$$

Let $T = \frac{1}{\sqrt{1+\alpha^2}}\left[\begin{smallmatrix} 1 & \alpha \\ \alpha & -1 \end{smallmatrix}\right]$, then $T = T^{\mathtt{T}} = T^{-1} \in \mathbf{O}_2(\mathbb{C})$ is orthogonal, and $R = TS = \left[\begin{smallmatrix} u & w \\ 0 & v \end{smallmatrix}\right]$ is upper triangular, where $v, w \in \mathbb{C}$ and $u = \sqrt{1+\alpha^2} \neq 0$. However, $\det R = \det T \det S = (-1)c \neq 0$, so we also have $v \neq 0$. It follows that

$$\begin{aligned}
T^{\otimes n} f &= (TS)^{\otimes n}[1,1,0,\ldots,0] \\
&= R^{\otimes n}[1,1,0,\ldots,0] \\
&= R^{\otimes n}\left([1,0]^{\otimes n} + \frac{1}{(n-1)!}\operatorname{Sym}_n^{n-1}([1,0];[0,1])\right) \\
&= [u,0]^{\otimes n} + \frac{1}{(n-1)!}\operatorname{Sym}_n^{n-1}([u,0];[w,v]) \\
&= [u^n + nu^{n-1}w, u^{n-1}v, 0, \ldots, 0].
\end{aligned}$$

Since $u^{n-1}v \neq 0$, we can normalize the entry of Hamming weight one to 1 by a scalar multiplication. Thus, we have $[z,1,0,\ldots,0]$ for some $z \in \mathbb{C}$.