Tensor topology

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informatics

“Where things happen”

- Any monoidal category comes with built-in ‘space’
- Matches examples
- Universal notion of support
- Completion to actual space
- Localisation to subspaces
- Embedding separates out spatial dimension

See also
[Balmer, “Tensor triangular geometry”]
[Boyarchenko&Drinfeld, “Character sheaves of unipotent groups”]

Idempotent subunits

Categorify central idempotents in ring:

$$\mathrm{ISub}(\mathbf{C}) = \left\{ s\colon S \rightarrowtail I \;\middle|\; S \otimes s\colon S \otimes S \to S \otimes I \text{ iso } \right\}$$

For most theory, split epic suffices
For simplicity, let's take $\mathbf{C}$ braided

Example: order theory

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e.g. open subsets of topological space

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$$
\begin{tikzcd}
\mathbf{Frame} \arrow[r, hook, shift left=2] \arrow[r, phantom, "\perp"] & \mathbf{Quantale} \arrow[l, shift left=2, "\mathrm{ISub}"] \\
\{x \in Q \mid x^2 = x \leq 1\} & Q \arrow[l, mapsto]
\end{tikzcd}
$$

Example: logic

$$\begin{aligned}\mathrm{ISub}(\mathrm{Sh}(X)) &= \{S \rightarrowtail 1\} \\ &= \{S \subseteq X \mid S \text{ open}\} \in \mathbf{Frame}\end{aligned}$$

Example: algebra

$$\mathrm{ISub}(\mathbf{Mod}_R) =$$

$$\left\{ S \subseteq R \text{ ideal} \;\middle|\; S = S^2 = \{x_1 y_1 + \cdots + x_n y_n \mid x_i, y_i \in S\} \right\}$$

for nonunital bialgebra R in monoidal category

Example: analysis

Hilbert module is $C_0(X)$-module with $C_0(X)$-valued inner product

$$C_0(X) = \{f \colon X \to \mathbb{C} \mid \forall \varepsilon > 0\ \exists K \subseteq X \colon |f(X \setminus K)| < \varepsilon\}$$

$$\mathrm{ISub}(\mathbf{Hilb}_{C_0(X)}) = \{S \subseteq X \text{ open}\}$$

Semilattice

Proposition: $\mathrm{ISub}(\mathbf{C})$ is a semilattice, $\wedge = \otimes$, $1 = I$

$$\begin{tikzcd}
T \arrow[dr, tail, "t"] & \\
& I \\
S \arrow[uu, dashed] \arrow[ur, tail, "s"'] &
\end{tikzcd}$$

Caveat: $\mathbf{C}$ must be firm, i.e. $s \otimes T$ monic, and size issue

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$$\begin{tikzcd} \mathbf{SemiLat} \arrow[loop left, "\mathrm{id}"] \arrow[r, dashed, shift left=2] \arrow[r, phantom, "\perp"] & \mathbf{FirmCat} \arrow[l, shift left=2, "\mathrm{ISub}"] \end{tikzcd}$$

Spatial categories

Call $\mathbf{C}$ spatial when $\mathrm{ISub}(\mathbf{C})$ is frame

$$\begin{tikzcd}
\mathbf{SemiLat} \arrow[r, shift left=1.5ex] \arrow[r, phantom, "\perp"] \arrow[d, shift right=1ex] \arrow[d, phantom, "\dashv"] & \mathbf{Frame} \arrow[l, hook', shift left=1.5ex] \arrow[d, shift right=1ex] \arrow[d, phantom, "\dashv"] \\
\mathbf{FirmCat} \arrow[u, shift right=1ex, "\mathrm{ISub}"'] \arrow[r, dashed, shift left=1.5ex, "?"] \arrow[r, phantom, "\perp"] & \mathbf{SpatCat} \arrow[l, hook', shift left=1.5ex] \arrow[u, shift right=1ex, "\mathrm{ISub}"']
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Idea: $\widehat{\mathbf{C}} = [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ is cocomplete

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Idea: $\widehat{\mathbf{C}} = [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ is cocomplete

$$F \widehat{\otimes} G(A) = \int^{B,C} \mathbf{C}(A, B \otimes C) \times F(B) \times G(C)$$

Lemma: $\mathrm{ISub}(\widehat{\mathbf{C}}, \widehat{\otimes})$ is frame

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Lemma: $\mathrm{ISub}(\widehat{\mathbf{C}}, \widehat{\otimes})$ is frame, but $\mathrm{ISub}(\widehat{\mathbf{C}}) \neq \widehat{\mathrm{ISub}(\mathbf{C})}$

Support

Say $s \in \mathrm{ISub}(\mathbf{C})$ supports $f \colon A \to B$ when

$$\begin{tikzcd} A \arrow[r, "f"'] \arrow[d, dashed] & B \\ B \otimes S \arrow[r, "B \otimes s"'] & B \otimes I \arrow[u, "\simeq"'] \end{tikzcd}$$

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$$\begin{tikzcd} f \arrow[r, mapsto] & \{s \mid s \text{ supports } f\} \\ \mathbf{C}^{\mathbf{2}} \arrow[r, "\mathrm{supp}"] & \mathrm{Pow}(\mathrm{ISub}(\mathbf{C})) \end{tikzcd}$$

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Monoidal functor: $\mathrm{supp}(f) \wedge \mathrm{supp}(g) \leq \mathrm{supp}(f \otimes g)$

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f \arrow[r, mapsto] & \{s \mid s \text{ supports } f\} \\
\mathbf{C^2} \arrow[r, "\mathrm{supp}"] \arrow[dr, "F"'] & \mathrm{Pow}(\mathrm{ISub}(\mathbf{C})) \arrow[d, dashed, "\widehat{F}"] \\
& Q \in \mathbf{Frame}
\end{tikzcd}$$

universal with $F(f) = \bigvee\{F(s) \mid s \in \mathrm{ISub}(\mathbf{C}) \text{ supports } f\}$

Spatial completion

Call $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ broad when

$$F(A) \simeq \big\{ (f, s)\colon A \to B \;\big|\; s \in \mathrm{supp}(f) \cap U \big\}$$

for some $B \in \mathbf{C}$ and $U \subseteq \mathrm{ISub}(\mathbf{C})$.

$$\begin{tikzcd}
\mathbf{SemiLat} \arrow[r, shift left=2] \arrow[r, phantom, "\perp"] \arrow[d, shift right=2] \arrow[d, phantom, "\dashv"] & \mathbf{Frame} \arrow[l, hook', shift left=2] \arrow[d, shift right=2] \arrow[d, phantom, "\dashv"] \\
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$$\mathbf{C} \longmapsto \hat{\mathbf{C}}_{\mathrm{brd}}$$

$\neq \mathrm{Sh}(\mathbf{C}, J)$!

Restriction

The full subcategory $\mathbf{C}\big|_s$ of A with $A \otimes s$ invertible is:

- monoidal with tensor unit S
- coreflective: $\mathbf{C}\big|_s \underset{\leftarrow - - - - -}{\overset{\hookrightarrow}{\perp}} \mathbf{C}$
- tensor ideal: if $A \in \mathbf{C}$ and $B \in \mathbf{C}\big|_s$, then $A \otimes B \in \mathbf{C}\big|_s$
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Examples: $(\mathbf{Mod}_R)\big|_I = \mathbf{Mod}_I$, $\mathrm{Sh}(X)\big|_U = \mathrm{Sh}(U)$

Localisation

A graded monad is a monoidal functor $\mathbf{E} \to [\mathbf{C}, \mathbf{C}]$

$(\eta \colon A \to T(1),\ \mu \colon T(t) \circ T(s) \to T(s \otimes t))$

Lemma: $s \mapsto \mathbf{C}\big|_s$ is an $\mathrm{ISub}(\mathbf{C})$-graded monad

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universal property of localisation for $\Sigma_s = \{A \otimes s \mid A \in \mathbf{C}\}$

$$\begin{tikzcd}
\mathbf{C} \arrow[rr, "(-) \otimes S"] \arrow[drr, "F \text{ inverting } \Sigma_s"'] & & \mathbf{C}\big|_s = \mathbf{C}[\Sigma_s^{-1}] \arrow[d, dashed] \arrow[dl, phantom, "\simeq"] \\
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Lemma: $\Sigma = \{A \otimes s \mid A \in \mathbf{C}, s \in \mathrm{ISub}(\mathbf{C}\}$ calculus of right fractions gives functor $\mathbf{C} \to \mathrm{Loc}(\mathbf{C}) = \mathbf{C}[\Sigma^{-1}]$ into simple category

Support structure

Say **C** is slim when any object is (domain of) idempotent subunit
(Note: S determines s)

Definition: support structure is functor $\zeta \colon \mathbf{C} \to \mathbf{C}$ with morphisms

- $\beta_A \colon \zeta(A) \rightarrowtail I$;
- $\gamma_A \colon A \to \zeta(A) \otimes A$;
- $\delta_A \colon \zeta(\zeta(A)) \to \zeta(A)$;

satisfying five coherence conditions

Proposition: δ_A is iso, β_A is idempotent, $\zeta \colon \mathbf{C} \to \mathrm{ISub}(\mathbf{C})$

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Proposition: δ_A is iso, β_A is idempotent, $\zeta\colon \mathbf{C} \to \mathrm{ISub}(\mathbf{C})$

Theorem: Any supported monoidal category embeds into product of simple and slim one: $\mathbf{C} \rightarrowtail \mathrm{Loc}(\mathbf{C}) \times \mathrm{ISub}(\mathbf{C})$

Conclusion

- Any monoidal category comes with built-in ‘space’
- Matches examples
- Universal notion of support
- Completion to actual space
- Localisation to subspaces
- Embedding separates out spatial dimension

Further goals:

- Canonical status for support structure
- Dauns-Hofmann-like theorem
- Graphical calculus
- Applications: causality, concurrency

Coherence

$$
\begin{tikzcd}
\zeta^2 A \arrow[r, "\delta"] \arrow[d, "\zeta\beta"'] \arrow[dr, "\beta"] & \zeta A \arrow[d, "\beta"] \\
\zeta I \arrow[r, "\beta"'] & I
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \zeta A \otimes A \arrow[dr, "\beta \otimes A"] & \\
A \arrow[ur, "\gamma"] \arrow[rr] & & I \otimes A
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & I \arrow[dr, "\gamma"] & \\
\zeta I \arrow[ur, "\beta"] \arrow[rr] & & \zeta I \otimes I
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & I \otimes \zeta^2 A \otimes \zeta A \arrow[dr, "I \otimes \delta \otimes \zeta A"] & \\
\zeta A \otimes \zeta A \arrow[ur, "\beta \otimes \gamma"] \arrow[rr] & & I \otimes \zeta \otimes \zeta A
\end{tikzcd}
$$

$$
\begin{tikzcd}
\zeta A \arrow[r, "\gamma"] \arrow[d, "\beta"'] & \zeta^2 A \otimes \zeta A \arrow[d, "\beta \otimes \beta"] \\
I \arrow[r] & I \otimes I
\end{tikzcd}
$$

Complements

Subunit is split when $\begin{tikzcd} S \arrow[loop left, "\mathrm{id}"] \arrow[r, shift left, tail, "s"] & I \arrow[l, shift left, dashed, two heads] \end{tikzcd}$

SISub($\mathbf{C}$) is a sub-semilattice of ISub($\mathbf{C}$)

(don't need firmness)

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If $\mathbf{C}$ has zero object, ISub($\mathbf{C}$) has least element 0

$s, s^{\perp}$ are complements if $s \wedge s^{\perp} = 0$ and $s \vee s^{\perp} = 1$

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Subunit is split when $\mathrm{id} \circlearrowright S \underset{\dashleftarrow}{\overset{s}{\rightarrowtail}} I$

$\mathrm{SISub}(\mathbf{C})$ is a sub-semilattice of $\mathrm{ISub}(\mathbf{C})$

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If $\mathbf{C}$ has zero object, $\mathrm{ISub}(\mathbf{C})$ has least element 0

$s, s^\perp$ are complements if $s \wedge s^\perp = 0$ and $s \vee s^\perp = 1$

Proposition: when $\mathbf{C}$ has finite biproducts,
then $s, s^\perp \in \mathrm{SISub}(\mathbf{C})$ are complements
if and only if they are biproduct injections

Corollary: if $\oplus$ distributes over $\otimes$,
then $\mathrm{SISub}(\mathbf{C})$ is a Boolean algebra
(universal property?)