Cloak and dagger

Chris Heunen

Algebra and coalgebra

Increasing generality:

- Vector space with bilinear (co)multiplication
- (Co)monoid in monoidal category
- (Co)monad: (co)monoid in functor category
- (Co)algebras for a (co)monad

Interaction between algebra and coalgebra?

Cloak and dagger

- Situation involving secrecy or mystery

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- Dagger, a concealable and silent weapon: dagger categories
- Cloak, worn to hide identity: Frobenius law

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- *Unitary representations*: $[G, \mathbf{Hilb}]_\dagger$

Never bring a knife to a gun fight

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  (but beats identity-on-objects-involutive-contravariant-functor)

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"Homotopy type theory"
Univalent foundations program, 2013

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- Dagger category theory different beast:
  isomorphism is not the correct notion of 'sameness'

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What about monoids??

Cloaks are worn

Frobenius algebra

Many definitions over a field k:

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- algebra A with equivalent left and right regular representations

"Theorie der hyperkomplexen Größen I"
Sitzungsberichte der Preussischen Akademie der Wissenschaften 504–537, 1903

"On Frobeniusean algebras II"
Annals of Mathematics 42(1):1–21, 1941

Frobenius law in algebra

Any finite group G induces Frobenius group algebra A:

- A has orthonormal basis $\{g \in G\}$
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- comultiplication $g \mapsto \sum_h gh^{-1} \otimes h$
- both sides of Frobenius law evaluate to $\sum_k gk^{-1} \otimes kh$ on $g \otimes h$

So Frobenius algebra incorporates finite group representation theory

Frobenius law in algebra

Frobenius algebras are wonderful:

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Frobenius law in algebra

Frobenius algebras are wonderful:

- left and right Artinian
- left and right self-injective
- Frobenius property is independent of base field k!
  - *Extension* of scalars: if l extends k, then
    A Frobenius over k iff $l \otimes_k A$ Frobenius over l
  - *Restriction* of scalars: if l extends k, then
    A Frobenius over l iff A Frobenius over k

Frobenius law in mathematics

- Number theory: commutative Frobenius algebras are Gorenstein

"Modular elliptic curves and Fermat's last theorem"
Annals of Mathematics 142(3):443–551, 1995

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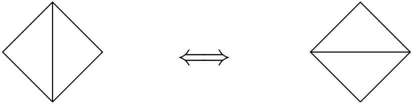
  Computes manifold invariants via Pachner moves:



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"P. L. homeomorphic manifolds are equivalent by elementary shellings"
European Journal of Combinatorics 12(2):129–145, 1991

Cloak

Dagger Frobenius structures: definition

In a dagger monoidal category: a dagger Frobenius structure consists of an object A and maps $\mu\colon A \otimes A \to A$ and $\eta\colon I \to A$ satisfying

$$(\mu \otimes \mathrm{id}) \circ \mu = (\mathrm{id} \otimes \mu) \circ \mu$$
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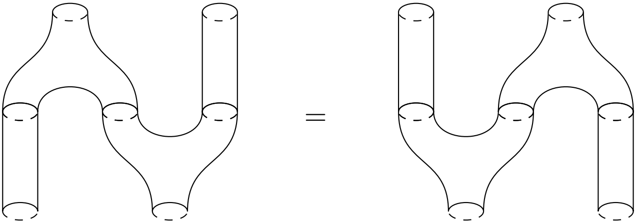
It can be:

- commutative: $\mu \circ \beta = \mu$ (in braided monoidal category)
- symmetric: $\eta^\dagger \circ \mu \circ \beta = \eta^\dagger \circ \mu$ (in braided monoidal category)
- special / strongly separable: $\mu \circ \mu^\dagger = \mathrm{id}$
- normalizable: $\mu \circ \mu^\dagger$ invertible, positive, and central

Frobenius algebra example: cobordisms

Category of cobordisms:

- objects are 1-dimensional compact manifolds
- arrows are 2-dimensional compact manifolds with boundary



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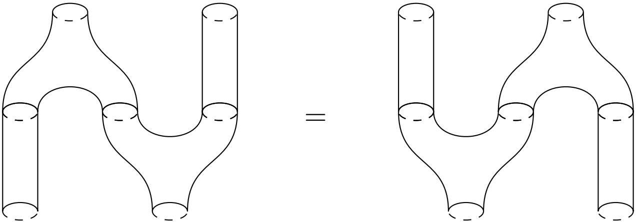

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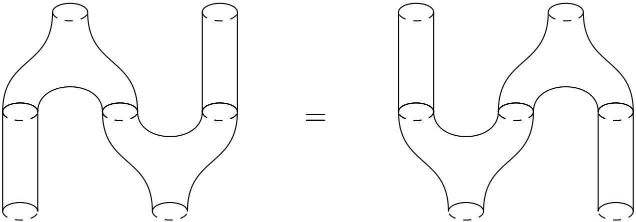

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(2d TQFT is just a monoidal functor $(\mathbf{Cob}, +) \to (\mathbf{FHilb}, \otimes)$)

"Frobenius algebras and 2D topological quantum field theories"
Cambridge University Press, 2003

Frobenius algebra example: C*-algebras

In the category of finite-dimensional Hilbert spaces:

- $\mathbb{M}_n$ is a monoid under $\mu\colon e_{ij} \otimes e_{kl} \mapsto \delta_{jk} e_{il}$
- $\mu^\dagger\colon e_{ij} \mapsto \sum_k e_{ik} \otimes e_{kj}$ satisfies Frobenius law:

$$e_{ij} \otimes e_{kl} \mapsto \delta_{jk} \sum_m e_{im} \otimes e_{ml}$$

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"Categorical formulation of finite-dimensional quantum algebras"
Communications in Mathematical Physics 304(3):765–796, 2011

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- in particular: commutative Frobenius structures are $\bigoplus_i \mathbb{M}_1$ that is, choice of orthonormal basis

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"A new description of orthogonal bases"
Mathematical Structures in Computer Science 23(3):555–567, 2013

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In the category of sets and relations:

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"Relative Frobenius algebras are groupoids"
Journal of Pure and Applied Algebra 217:114–124, 2013

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- conversely: all dagger Frobenius structures are groupoids



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Cloak hides dagger

Graphical calculus

"Notation which is useful in private must be given a public value and that it should be provided with a firm theoretical foundation"


"The geometry of tensor calculus I"
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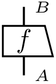
- Morphisms $f\colon A \to B$ depicted as boxes 

- Composition: stack boxes vertically
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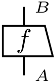
- Morphisms $f\colon A \to B$ depicted as boxes 

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Coherence isomorphisms melt away


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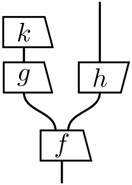
Sound: isotopic diagrams represent equal morphisms



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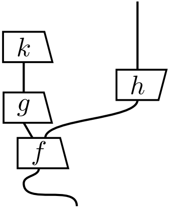

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New Structures for Physics, LNP 813:289–355, 2011

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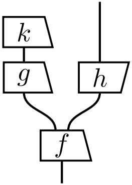

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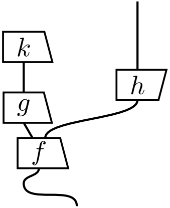

Complete: diagrams isotopic iff equal in category of Hilbert spaces

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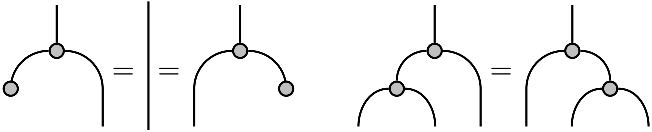
"Finite-dimensional Hilbert spaces are complete for dagger compact categories"
Logical Methods in Computer Science 8(3:6):1–12, 2012

Frobenius law graphically

Instead of box, will draw  for multiplication $A \otimes A \to A$ of monoid.



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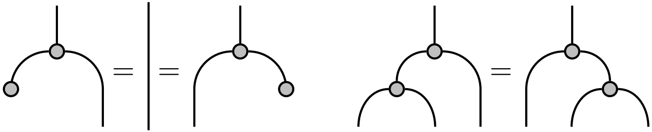
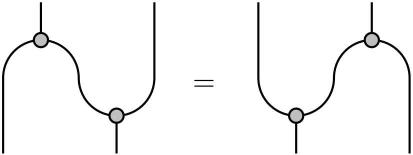


Frobenius law becomes:

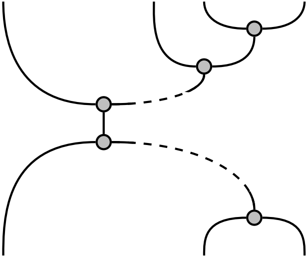


“Ordinal sums and equational doctrines”
Seminar on triples and categorical homology theory, LNCS 80:141–155, 1966

“Two-dimensional topological quantum field theories and Frobenius algebras”
Journal of Knot Theory and its Ramifications 5:569–587, 1996

Spider theorem

Any connected diagram built from the components of a special ( = |) Frobenius algebra equals the following normal form:



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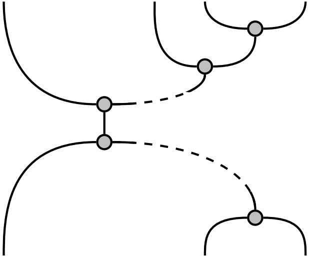

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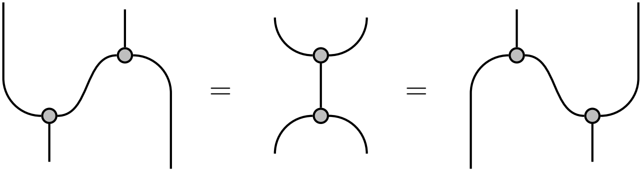
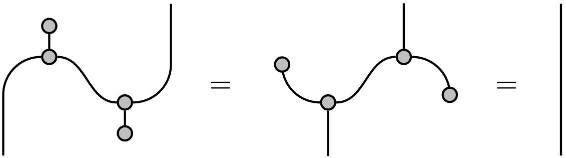


Dual objects

Note:



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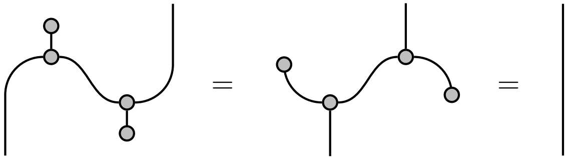
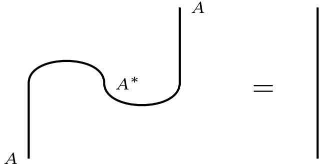


Hence any Frobenius structure is self-dual



Dual objects: examples

In category of finite-dimensional Hilbert spaces:

- Canonical duals: $\smile\colon \mathbb{C} \to H \otimes H^*$ given by $1 \mapsto \sum_i e_i \otimes e_i^*$

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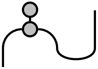
- Involution  on $\mathbb{M}_n$ is precisely $a \mapsto a^\dagger$

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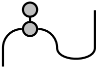
- Involution  on $\mathbb{M}_n$ is precisely $a \mapsto a^\dagger$

In category of sets and relations:

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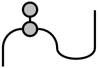
- Involution  on $\mathbb{M}_n$ is precisely $a \mapsto a^\dagger$

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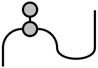
- Involution  on $\mathbb{M}_n$ is precisely $a \mapsto a^\dagger$

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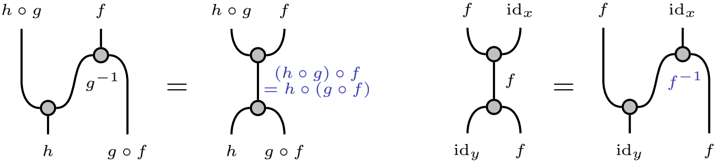
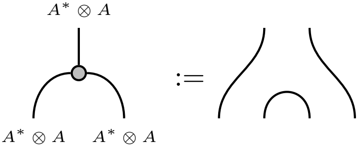
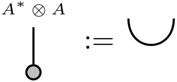


Pairs of pants

If an object A has a dual, then $A^* \otimes A$ is a monoid





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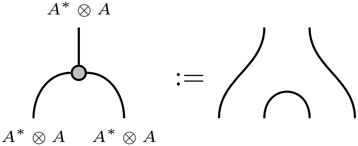
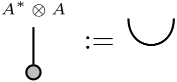
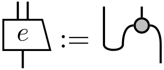
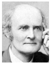




Any monoid (A, ) embeds into $A^* \otimes A$ by 



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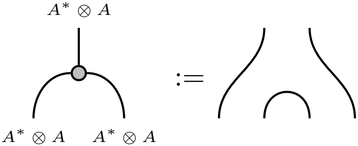
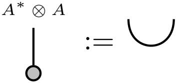
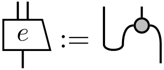
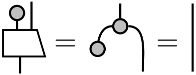
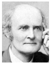




Any monoid (A, ) embeds into $A^* \otimes A$ by 

- is embedding: 



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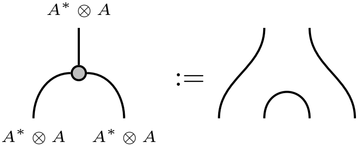
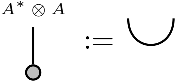
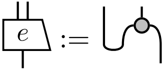
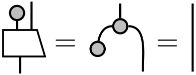
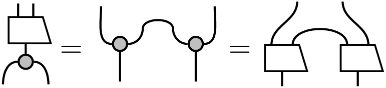
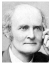




Any monoid (A, ) embeds into $A^* \otimes A$ by 

- is embedding: 

- preserves multiplication: 



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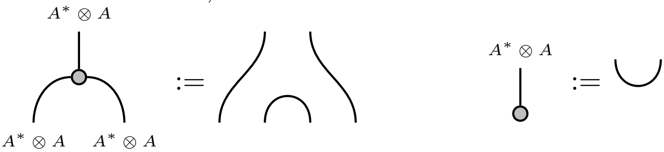
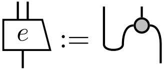
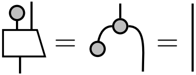
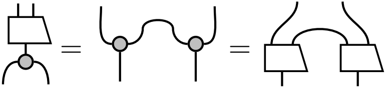
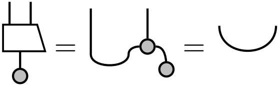
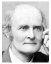


Any monoid (A, ) embeds into $A^* \otimes A$ by 

- is embedding: 
- preserves multiplication: 
- preserves unit: 



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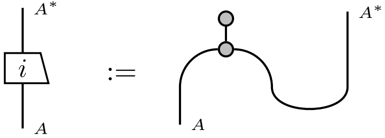
Why the Frobenius law?

Maps $I \to A^* \otimes A$ correspond to maps $A \to A$.

So in presence of dagger, $A^* \otimes A$ is involutive monoid!



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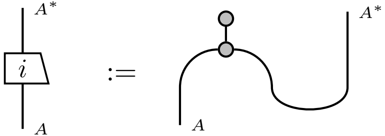

Theorem in a monoidal dagger category:
a monoid is Frobenius $\Leftrightarrow$ its Cayley embedding is involutive


“Reversible monadic computing”
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“Geometry of abstraction in quantum computation”
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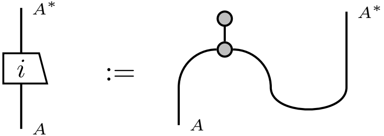

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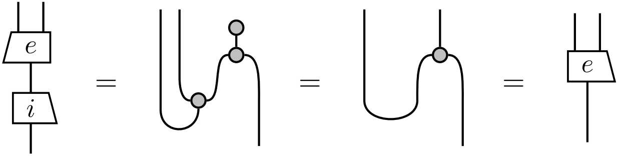


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Frobenius monads

- Let **C** be a monoidal category
- A monad is a monoid in $[\mathbf{C}, \mathbf{C}]$
- A monad T on a **C** is strong when equipped with a natural transformation $A \otimes T(B) \to T(A \otimes B)$
- **Theorem**: There is an adjunction between monoids in **C** and strong monads on **C**.

$$A \mapsto - \otimes A$$
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Frobenius monads

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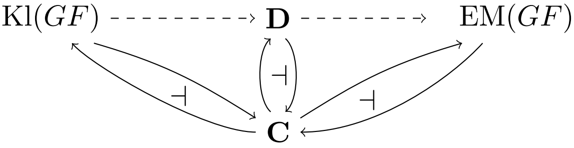
Algebras

Let $\mathbf{C}$ and $\mathbf{D}$ be categories, $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$ be functors with $F \dashv G$. Then GF is monad with:



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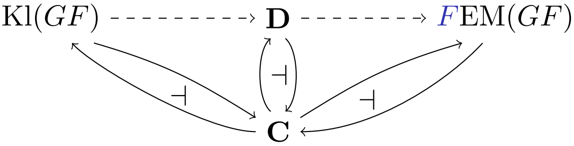

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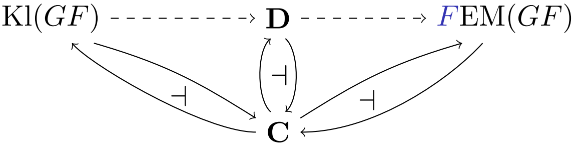

Conversely, if a monad on **C** is Frobenius then it is of this form.

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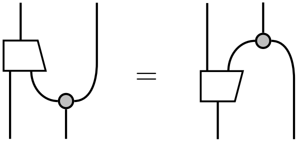

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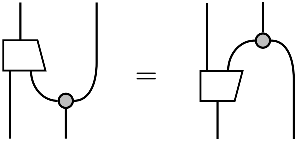

They form the largest subcategory of $\mathrm{EM}(T)$ that inherits dagger.

Summary

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- "Dagger likes cloak": Frobenius monads are dagger adjunctions (free) algebra categories again have dagger

Quantum measurement

Fix orthonormal basis on $\mathbb{C}^n$ so $T = - \otimes \mathbb{C}^n$ is Frobenius monad on category of Hilbert spaces. Measurement is map $A \to T(A)$.

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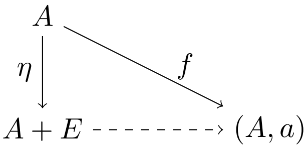
Consider exception monad $T = - + E$



- intercept exception e: execute f_e, or f if no exception
- handler for T specifies EM-algebra (A, a) and $f \colon A \to A$
- vertical arrows are Kleisli maps, dashed one EM-map

“Classical and quantum structuralism”
Semantic Techniques in Quantum Computation 29–69, 2009

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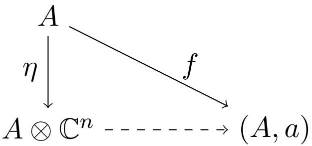

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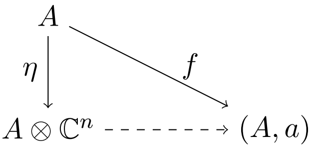

- Kleisli maps $A \to T(B)$ 'build' effectful computation
- FEM-algebras $T(B) \to B$ are destructors 'handling' the effects
- Effectful computation for Frobenius monad happens in FEM(T)

"Classical and quantum structuralism"
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