Domains of commutative C*-subalgebras

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Chris Heunen and Bert Lindenhovius 

Logic in Computer Science 2015

EPSRC Pioneering research and skills
UNIVERSITY OF OXFORD DEPARTMENT OF COMPUTER SCIENCE
QMAC
St Hugh's College UNIVERSITY OF OXFORD

Measurement

State: unit vector x in $\mathbb{C}^n$

Measurement: in basis $e_1, \ldots, e_n$
gives outcome i with probability $\langle e_i \mid x\rangle$

Measurement

State: unit vector x in $\mathbb{C}^n$

Measurement: hermitian matrix e in $\mathbb{M}_n$ with eigenvectors e_i
given by $|i\rangle \mapsto |e_i\rangle\langle e_i|$
gives outcome i with probability $\langle e_i \mid x\rangle$

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Measurement: hermitian matrix e in $\mathbb{M}_n$
given by $|i\rangle \mapsto |e_i\rangle\langle e_i|$
gives outcome i with probability $\mathrm{tr}(|e_i\rangle\langle e_i|x)$

Measurement

State: unit vector x in $\mathbb{C}^n$

Measurement: function $e\colon \mathbb{C}^n \to \mathbb{M}_n$ such that
- e linear
- $e(1,\ldots,1) = 1$
- $e(x_1y_1,\ldots,x_ny_n) = e(x)e(y)$
- $e(\overline{x_1},\ldots,\overline{x_n}) = e(x)^*$

gives outcome i with probability $\mathrm{tr}(e|i\rangle\, x)$

Measurement

State: unit vector x in $\mathbb{C}^n$

Measurement: unital $*$-homomorphism $e\colon \mathbb{C}^n \to \mathbb{M}_n$
gives outcome i with probability $\mathrm{tr}(e|i\rangle\, x)$

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State: unit vector x in Hilbert space H

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"projection-valued measure" (PVM)
"sharp measurement"

Compatible measurements

PVMs $e, f\colon \mathbb{C}^m \to B(H)$ are jointly measurable
when each $e|i\rangle$ and $f|j\rangle$ commute.

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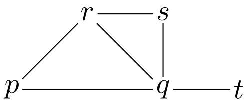
(In)compatibilities form graph:



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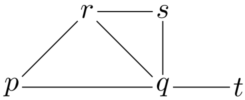

Theorem: Any graph can be realised as PVMs on a Hilbert space.

  

"Quantum theory realises all joint measurability graphs"
Physical Review A 89(3):032121, 2014

Probabilistic measurement

State: unit vector x in Hilbert space H

Measurement: function $e\colon \mathbb{C}^m \to B(H)$ such that
- e linear
- $e(1, \ldots, 1) = 1$
- $e(x) \geq 0$ if all $x_i \geq 0$

gives outcome i with probability $\mathrm{tr}(e|i\rangle\, x)$

Probabilistic measurement

State: unit vector x in Hilbert space H

Measurement: function $e \colon \mathbb{C}^m \to B(H)$ such that
- e linear
- $e(1, \ldots, 1) = 1$
- $e(x_1^* x_1, \ldots, x_n^* x_n) = a^* a$ for some a in $B(H)$

gives outcome i with probability $\mathrm{tr}(e|i\rangle\, x)$

Probabilistic measurement

State: unit vector x in Hilbert space H

Measurement: unital (completely) positive linear $e\colon \mathbb{C}^m \to B(H)$
gives outcome i with probability $\mathrm{tr}(e|i\rangle\, x)$

Probabilistic measurement

State: unit vector x in Hilbert space H

Measurement: unital (completely) positive linear $e\colon \mathbb{C}^m \to B(H)$
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"positive-operator valued measure" (POVM)
"unsharp measurement"

Compatible probabilistic measurements

POVMs $e, f\colon \mathbb{C}^m \to B(H)$ are jointly measurable
when there exists POVM $g\colon \mathbb{C}^{m^2} \to B(H)$
such that $e|i\rangle = \sum_j g|ij\rangle$ and $f|j\rangle = \sum_i g|ij\rangle$
(e, f are *marginals* of g)

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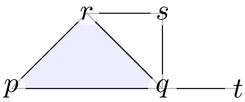
(In)compatibilities form hypergraph:



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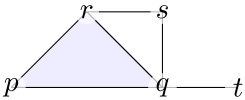
(e, f are *marginals* of g)

(In)compatibilities form abstract simplicial complex:



Theorem: Any abstract simplicial complex can be realised as POVMs on a Hilbert space.



"All joint measurability structures are quantum realizable"
Physical Review A 89(5):052126, 2014

States

State: unit vector x in Hilbert space H

Measurement: unital (completely) positive linear $e\colon \mathbb{C}^m \to B(H)$
gives outcome i with probability $\mathrm{tr}(e|i\rangle\, x)$

States

State: ensemble of
unit vectors x in Hilbert space H

Measurement: unital (completely) positive linear $e\colon \mathbb{C}^m \to B(H)$
gives outcome i with probability $\mathrm{tr}(e|i\rangle\, x)$

States

State: ensemble of
projections $|x\rangle\langle x|$ onto vectors in Hilbert space H

Measurement: unital (completely) positive linear $e\colon \mathbb{C}^m \to B(H)$
gives outcome i with probability $\mathrm{tr}(e|i\rangle\, |x\rangle\langle x|)$

States

State: ensemble of
rank one projections $p^2 = p = p^*$ in $B(H)$

Measurement: unital (completely) positive linear $e\colon \mathbb{C}^m \to B(H)$
gives outcome i with probability $\mathrm{tr}(e|i\rangle\,|x\rangle\langle x|)$

States

State: positive operator ρ in $B(H)$ of norm 1

Measurement: unital (completely) positive linear $e\colon \mathbb{C}^m \to B(H)$
gives outcome i with probability $\mathrm{tr}(e|i\rangle\,\rho)$

States

State: linear function $\rho\colon B(H) \to \mathbb{C}$ such that
$\rho(a) \geq 0$ if $a \geq 0$, and $\rho(1) = 1$

Measurement: unital (completely) positive linear $e\colon \mathbb{C}^m \to B(H)$
gives outcome i with probability $\mathrm{tr}(e|i\rangle\,\rho)$

States

State: unital (completely) positive linear $\rho \colon B(H) \to \mathbb{C}$
"density matrix"

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So really only the set $B(H)$ matters.
It is a C*-algebra.

States

State: unital (completely) positive linear $\rho\colon A \to \mathbb{C}$
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The above works for *any* C*-algebra A:
can formulate measurements, and derive states in terms of A alone

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So really only the set $B(H)$ matters.
It is a noncommutative C*-algebra.

The above works for *any* C*-algebra A:
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Continuous measurement

State: unital (completely) positive linear $\rho\colon A \to \mathbb{C}$

Measurement: with m discrete outcomes
unital (completely) positive linear $e\colon \mathbb{C}^m \to A$

Continuous measurement

State: unital (completely) positive linear $\rho\colon A \to \mathbb{C}$

Measurement: with outcomes in compact Hausdorff space X
unital (completely) positive linear $e\colon C(X) \to A$

Here, $C(X) = \{f\colon X \to \mathbb{C} \text{ continuous}\}$ is a commutative C*-algebra.

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Theorem: Every commutative C*-algebra is of the form $C(X)$.

"On normed rings"
Doklady Akademii Nauk SSSR 23:430–432, 1939

Classical data

Unsharp measurement: unital positive linear $e\colon C(X) \to A$
Sharp measurement: unital $*$-homomorphism $e\colon C(X) \to A$

Measurement: only way to get (classical) data from quantum system

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Theorem: 'unsharp measurements can be dilated to sharp ones': any POVM $e\colon C(X) \to B(H)$ allows a PVM $f\colon C(X) \to B(K)$ and isometry $v\colon H \to K$ such that $e(-) = v^* \circ f(-) \circ v$.

Sharp measurements give all (accessible) data about quantum system

"Positive functions on C*-algebras"
Proceedings of the American Mathematical Society, 6(2):211–216, 1955

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Sharp measurements give all (accessible) data about quantum system

Lemma: the image of a unital $*$-homomorphism $e\colon C(X) \to A$ is a (unital) commutative C*-subalgebra of A.

Commutative C*-subalgebras record all data of quantum system

"Positive functions on C*-algebras"
Proceedings of the American Mathematical Society, 6(2):211–216, 1955

Coarse graining

Can collapse measurement with 3 outcomes into measurement with 2 outcomes by pretending two states are the same.

continuous function $X \to Y$	$\rightsquigarrow$	$*$-homomorphism $C(Y) \to C(X)$
surjection $X \twoheadrightarrow Y$	$\rightsquigarrow$	injection $C(Y) \rightarrowtail C(X)$
quotient of state space X	$\rightsquigarrow$	C*-subalgebra of $C(X)$

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Larger C*-subalgebras give more information

going up in order = better classical approximations (tomography)

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Larger C*-subalgebras give more information
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Definition: If A is a C*-algebra, $\mathcal{C}(A)$ is the set of commutative C*-subalgebras, partially ordered by inclusion $\subseteq$.

Results about $\mathcal{C}(A)$: topos

- Consider "contextual sets" over C*-algebra A
  assignment of set $S(C)$ to each $C \in \mathcal{C}(A)$
  such that $C \subseteq D$ implies $S(C) \rightarrowtail S(D)$

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- There is one canonical contextual set $\underline{A}$
  $\underline{A}(C) = C$

- $\mathcal{T}(A)$ believes that $\underline{A}$ is a commutative C*-algebra!



"A Topos for Algebraic Quantum Theory"
Communications in Mathematical Physics 291:63–110, 2009

Results about $\mathcal{C}(A)$: reconstruction

To what extent does $\mathcal{C}(A)$ determine A?

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"Characterizations of Categories of Commutative C*-subalgebras"
Communications in Mathematical Physics 331(1):215–238, 2014

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If $\mathcal{C}(A) \cong \mathcal{C}(B)$, then $A \cong B$ as Jordan algebras.
(Except $\mathbb{C}^2$ and $\mathbb{M}_2$.)

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"Abelian Subalgebras and Jordan Structure of Von Neumann Algebras"
Houston Journal of Mathematics, 2015

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Journal of Mathematical Analysis and Applications 383:391–399, 2011

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If $\mathcal{C}(A) \cong \mathcal{C}(B)$ and A finite-dimensional, then $A \cong B$.

"Characterizations of Categories of Commutative C*-subalgebras"
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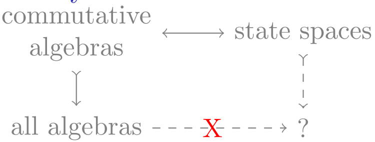
Journal of Mathematical Analysis and Applications 383:391–399, 2011

"Classifying fininite-dim'l C*-algebras by posets of commutative C*-subalgebras"
International Journal of Theoretical Physics, 2015

Non-results about $\mathcal{C}(A)$: reconstruction

Extra ingredient necessary to reconstruct A:



"Extending Obstructions to Noncommutative Functorial Spectra"
Theory and Applications of Categories 29(17):457–474, 2014

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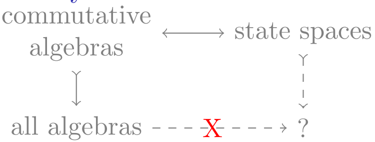

Trace *almost* suffices as extra ingredient.

(If associative $*\colon \mathbb{M}_n \otimes \mathbb{M}_n \to \mathbb{M}_n$ satisfies $xy = yx \implies x * y = xy$ and $\mathrm{Tr}(x * y) = \mathrm{Tr}(xy)$, then it must be matrix multiplication *(or opposite)*.)

 

"Extending Obstructions to Noncommutative Functorial Spectra"
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"Matrix Multiplication is determined by Orthogonality and Trace"
Linear Algebra and its Applications 439(12):4130–4134, 2013

Non-results about $\mathcal{C}(A)$: reconstruction

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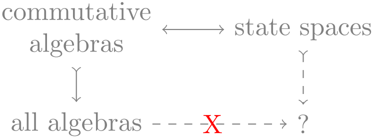

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Orientation suffices as extra ingredient.

(If $\mathcal{C}(A) \cong \mathcal{C}(B)$ preserves $U(A) \times \mathcal{C}(A) \to \mathcal{C}(A)$ then $A \cong B$.)

 

"Extending Obstructions to Noncommutative Functorial Spectra"
Theory and Applications of Categories 29(17):457–474, 2014

 

"Matrix Multiplication is determined by Orthogonality and Trace"
Linear Algebra and its Applications 439(12):4130–4134, 2013

 

"Active Lattices determine AW*-algebras"
Journal of Mathematical Analysis and Applications 416:289-313, 2014

What kind of partial order is $\mathcal{C}(A)$?

Lemma: Chains C_i in $\mathcal{C}(A)$ have least upper bound $\bigvee C_i := \overline{\bigcup C_i}$.

May regard A as 'ideal' system approximated by C_i.

What kind of partial order is $\mathcal{C}(A)$?

Lemma: Chains C_i in $\mathcal{C}(A)$ have least upper bound $\bigvee C_i := \overline{\bigcup C_i}$.

May regard A as 'ideal' system approximated by C_i.

Common refinement:

Lemma: Nonempty $\{C_i\}$ have greatest lower bound $\bigwedge C_i := \bigcap C_i$.

"The space of measurement outcomes as a spectral invariant"
Foundations of Physics 42:896–908, 2012

Domains

Desirable properties:

- Continuous: can take approximants way below
  $C = \bigvee \{ B \mid C \leq \bigvee B_i \implies \exists i \colon B \leq B_i \}$


“Domain Theory”
Handbook of Logic in Computer Science 3, 1994





“Continuous Lattices and Domains”
Cambridge University Press, 2003

Domains

Desirable properties:

- Continuous: can take approximants way below
  $C = \bigvee\{B \mid C \leq \bigvee B_i \implies \exists i \colon B \leq B_i\}$
- Algebraic: can take approximants compact
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- Algebraic: can take approximants compact
  $C = \bigvee\{B \leq C \mid B \leq \bigvee B_i \implies \exists i \colon B \leq B_i\}$
- Quasi-continuous: finitely many observations per approximant
- Quasi-algebraic: finitely many observations per approximant


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- Quasi-continuous: finitely many observations per approximant

- Quasi-algebraic: finitely many observations per approximant

- Atomistic: approximation proceeds in indivisible steps
  $C = \bigvee\{B > 0 \mid 0 < B' \leq B \implies B' = B\}$


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- Quasi-continuous: finitely many observations per approximant
- Quasi-algebraic: finitely many observations per approximant
- Atomistic: approximation proceeds in indivisible steps
  $C = \bigvee\{B > 0 \mid 0 < B' \leq B \implies B' = B\}$
- Meet-continuous: approximation respects restriction
  $C \wedge \bigvee C_i = \bigvee C \wedge C_i$


“Domain Theory”
Handbook of Logic in Computer Science 3, 1994





“Continuous Lattices and Domains”
Cambridge University Press, 2003

Robust approximation

Theorem: For a C*-algebra A, the following are equivalent:

- $\mathcal{C}(A)$ is continuous;
- $\mathcal{C}(A)$ is algebraic;
- $\mathcal{C}(A)$ is quasi-continuous;
- $\mathcal{C}(A)$ is quasi-algebraic;
- $\mathcal{C}(A)$ is atomistic;
- $\mathcal{C}(A)$ is meet-continuous;


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- $\mathcal{C}(A)$ is atomistic;
- $\mathcal{C}(A)$ is meet-continuous;
- A is scattered


"Domains of commutative C*-subalgebras"
Logic in Computer Science, 2015

Degeneration

Could play same game with von Neumann algebras A,
with commutative von Neumann subalgebras $\mathcal{V}(A) = \{C \subseteq A\}$.

Proposition: For W*-algebras A there is a Galois correspondence:

$$\mathcal{V}(M) \underset{\hookrightarrow}{\overset{\longleftarrow}{\bot}} \mathcal{C}(M)$$

Degeneration

Could play same game with von Neumann algebras A,
with commutative von Neumann subalgebras $\mathcal{V}(A) = \{C \subseteq A\}$.

Proposition: For W*-algebras A there is a Galois correspondence:

$$\mathcal{V}(M) \underset{\hookrightarrow}{\overset{\longleftarrow}{\perp}} \mathcal{C}(M)$$

However, von Neumann algebras are rarely scattered.

Theorem: The following are equivalent for W*-algebras A:

- $\mathcal{C}(A)$ is continuous
- $\mathcal{C}(A)$ is algebraic
- $\mathcal{V}(A)$ is continuous
- $\mathcal{V}(A)$ is algebraic
- A is finite-dimensional


"Unsharp values, domains and topoi"
Quantum field theory and gravity 65–96, 2012

Algebraic approximation

Can only access finite-dimensional subalgebras in finite time.

Definition: A C*-algebra A is approximately finite-dimensional when $A = \overline{\bigcup A_i}$ for a chain A_i of finite-dimensional C*-algebras.

"Inductive Limits of Finite Dimensional C*-algebras"
Transactions of the American Mathematical Society 171:195–235, 1972

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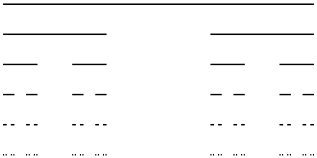
Definition: A C*-algebra A is approximately finite-dimensional when $A = \overline{\bigcup A_i}$ for a chain A_i of finite-dimensional C*-algebras.

- If $X = [0, 1]$, then $C(X)$ is not approximately finite-dimensional
- If X is Cantor set, $C(X)$ is approximately finite-dimensional



"Inductive Limits of Finite Dimensional C*-algebras"
Transactions of the American Mathematical Society 171:195–235, 1972

Scatteredness

Definition: A topological space is scattered if every nonempty closed subset has an isolated point.

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Definition: A C*-algebra A is scattered when, equivalently:

- each $C \in \mathcal{C}(A)$ is approximately finite-dimensional
- X is scattered for each maximal $C(X) \in \mathcal{C}(A)$
- each state is a countable sum of pure ones

Example: the unitization of compact operators $K(H) + \mathbb{C}1_H$

"Scattered C*-algebras"
Mathematica Scandinavica 41:308–314, 1977

Higher order approximation

Topologies on $\mathcal{C}(A)$ whose notion of limit is that of approximation:

- Scott topology: if $f\colon A \to B$ is a $*$-homomorphism, then $\mathcal{C}(f)\colon \mathcal{C}(A) \to \mathcal{C}(B)$ is Scott continuous.
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Can speak about approximation within language of C*-algebras!
What is the relationship between A and $C(X)$?

- $A \mapsto X$ is not functorial
- No iteration: if A is scattered, then $\mathcal{C}(A)$ is scattered only if A is finite-dimensional

Labelled Transition Systems: deterministic

Model computational behaviour of discrete systems
e.g. traffic light, computer programs

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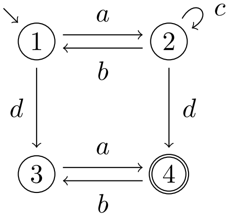

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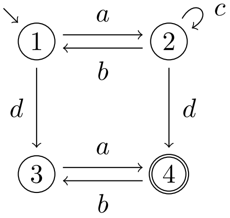

states: one at a time

transitions: move token

initial: place token

final: accept token

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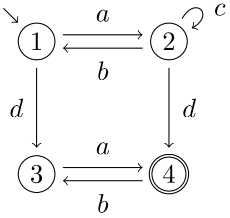

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transition matrices

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

entries in $\{0,1\}$
1 at (i,j) iff $i \xrightarrow{a} j$

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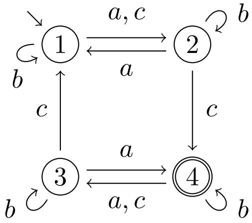

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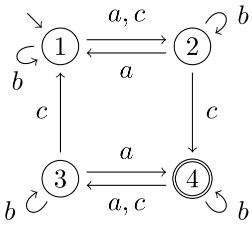

states: one at a time

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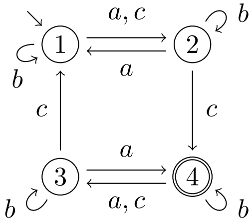

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permutation matrices

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

entries in $\{0, 1\}$
one 1 per row/column

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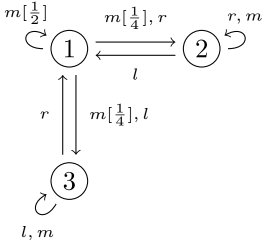

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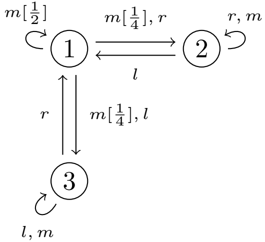

states: convex weights

transitions: stochastic

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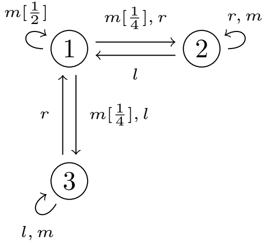

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stochastic matrices

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

entries in $[0, 1]$
rows sum to 1

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Model computational behaviour of quantum-mechanical systems
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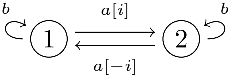

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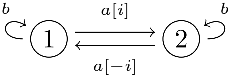

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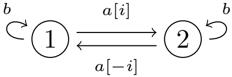

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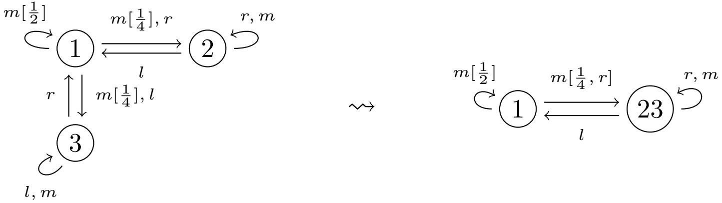
hermitian matrices

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

entries in $\mathbb{C}$

Approximating Labelled Transition Systems

Identify (bisimilar) states:



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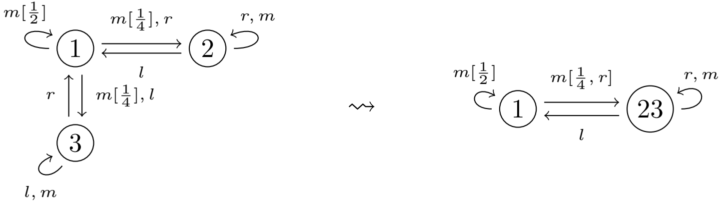

Invertible $\subsetneq$ Deterministic $\subsetneq$ Probabilistic $\subsetneq$ Quantum

Algebraic dualisation

Linking transitions	$\rightsquigarrow$	multiplying transition matrices
Reversing transitions	$\rightsquigarrow$	transposing transition matrices

Algebraic dualisation

Linking transitions $\rightsquigarrow$ multiplying transition matrices
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All possible runs $\rightsquigarrow$ algebra generated by transition matrices
(subset $\mathbb{M}_n$ closed under addition, multiplication, transpose)

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State space X	$\rightsquigarrow$	C*-algebra $C(X) = \{f\colon X \to \mathbb{C}\}$

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Quotient $\rightsquigarrow$ subalgebra



"Minimization via duality"
LNCS 7456:191–205, WoLLIC 2012

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Warning: different terminology states
Warning: duality up to trace semantics
Nevertheless: approximate transition system
commutative sublanguage?



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Conclusion

Questions:

- Approximate transition systems
- Universal construction $C(\mathcal{C}(A))$
- Solve domain equations
- Recognize structure of A from $\mathcal{C}(A)$ (e.g. postliminal, AW*)