# Coalgebraic Components in a Many-Sorted Microcosm

Ichiro Hasuo[1,4], Chris Heunen[2], Bart Jacobs[2], and Ana Sokolova[3⋆]

[1] RIMS, Kyoto University, Japan
[2] Radboud University Nijmegen, the Netherlands
[3] University of Salzburg, Austria
[4] PRESTO Research Promotion Program, Japan Science and Technology Agency

**Abstract.** The *microcosm principle*, advocated by Baez and Dolan and formalized for Lawvere theories lately by three of the authors, has been applied to coalgebras in order to describe compositional behavior systematically. Here we further illustrate the usefulness of the approach by extending it to a many-sorted setting. Then we can show that the coalgebraic component calculi of Barbosa are examples, with compositionality of behavior following from microcosm structure. The algebraic structure on these coalgebraic components corresponds to variants of Hughes' notion of *arrow*, introduced to organize computations in functional programming.

## 1 Introduction

Arguably the most effective countermeasure against today's growing complexity of computer systems is *modularity*: one should be able to derive the behavior of the total system from that of its constituent parts. Parts that were developed and tested in isolation can then safely be composed into bigger systems. Likewise, one would like to be able to prove statements about the compound system based on proofs of substatements about the parts. Therefore, the theoretical models should at the very least be such that their behavior is compositional.

This is easier said than done, especially in the presence of concurrency, that is, when systems can be composed in parallel as well as in sequence. The *microcosm principle* [1, 14] brings some order to the situation. Roughly speaking, compositionality means that the behavior of a compound system is the composition of the components' behaviors. The microcosm principle then observes that the very definition of composition of behaviors depends on composition of systems, providing an intrinsic link between the two.

The present article gives a rigorous analysis of compositionality of components as sketched above. Considering models as *coalgebras*, we study Barbosa's calculi of *components* [2, 3] as coalgebras with specified input and output interfaces. Explicitly, a component is a coalgebra for the endofunctor[5]

$$F_{I,J} = (T(J \times \_))^I : \mathbf{Set} \to \mathbf{Set}, \tag{1}$$

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[5] Note that the functor $F_{I,J}$ also depends on the choice of the parameter $T$, so could have been denoted e.g. by $F^T_{I,J}$. We shall not do so for simplicity of presentation.

where $I$ is the set of possible input, and $J$ that of output. The computational effect of the component is modeled by a monad $T$, as is customary in functional programming [25]. The monad $T$ can capture features such as finite or unbounded non-determinism ($T = \mathcal{P}_\omega, \mathcal{P}$); possible non-termination ($T = 1 + \_$); global states ($T = (S \times \_)^S$); probabilistic branching or combinations of these.

To accommodate component calculi, the surrounding microcosm needs to be *many-sorted*. After all, composing components sequentially requires that the output of the first and the input of the second match up. This is elaborated on in §2. The contribution of the present article is twofold:

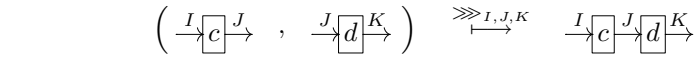
- a rigorous development of a many-sorted microcosm principle, in §4;
- an application of the many-sorted microcosm framework to component calculi, in §5.

It turns out that components as $F_{I,J}$-coalgebras carry algebraic structure that is a variant of Hughes' notion of *arrow* [15, 20].[6] Arrows, generalizing monads, have been used to model structured computations in semantics of functional programming. In §5 we will prove that components indeed carry such arrow-like structure; however the calculation is overwhelming as it is. To aid the calculation, we exploit the fact that a Kleisli category $\mathcal{K}\ell(T)$—where the calculation takes place—also carries the same arrow-like structure. This allows us to use the axiomatization of the (shared) structure as an "internal language."

## 2 Leading example: sequential composition

We shall exhibit, using the following example, the kind of phenomena in component calculi that we are interested in.

For simplicity we assume that we have no effect in components (i.e. $T = \mathrm{Id}$, $F_{I,J} = (J \times \_)^I$). Coalgebras for this functor are called *Mealy machines*, see e.g. [8]. A prominent operation in component calculi is *sequential composition*, or *pipeline*. It attaches two components with matching I/O interfaces, one after another:




(2)

Let $X$ and $Y$ be the state spaces of the components $c$ and $d$, respectively. The resulting component $c \ggg_{I,J,K} d$ has the state space $X \times Y$;[7] given input $i \in I$, first $c$ produces output $j \in J$ that is fed into the input port of $d$. More precisely, we can define the coalgebra $c \ggg_{I,J,K} d$ to be the adjoint transpose $X \times Y \to (K \times X \times Y)^I$ of the following function.

$$I \times X \times Y \xrightarrow{\hat{c} \times d} J \times X \times (K \times Y)^J \xrightarrow{X \times \mathsf{ev}_J} K \times X \times Y \tag{3}$$

Here $\hat{c} : I \times X \to J \times X$ is the adjoint transpose of the coalgebra $c$, and $\mathsf{ev}_J : J \times (K \times Y)^J \to K \times Y$ is the obvious evaluation function.

[6] Throughout the paper the word "arrow" always refers to Hughes' notion. An "arrow" in a category (as opposed to an object) will be always called a *morphism*.

[7] We will use the infix notation for the operation $\ggg$. The symbol $\ggg$ is taken from that for (Hughes') arrows, whose relevance is explained in §5.

An important ingredient in the theory of coalgebra is "behavior-by-coinduction" [19]: when a state-based system is viewed as an $F$-coalgebra, then a *final* $F$-coalgebra (which very often exists) consists of all the "behaviors" of systems of type $F$. Moreover, the morphism induced by finality is the "behavior map": it carries a state of a system to its behavior. This view is also valid in the current example.

$$\begin{array}{ccc} FX & \dashrightarrow & FZ \\ {\scriptstyle c}\uparrow & & \text{final}\uparrow\cong \\ X & \xrightarrow[\mathsf{beh}(c)]{} & Z \end{array}$$

A final $F_{I,J}$-coalgebra—where $F_{I,J} = (J \times \_)^I$—is carried by the set of stream functions $I^\omega \to J^\omega$ which are *causal*, meaning that the $n$-th letter of the output stream only depends on the first $n$ letters of the input.[8] It conforms to our intuition: the "behavior" of such a component is what we see as an output stream when an input stream is fed. The final coalgebra is concretely as follows.

$$\begin{array}{rccc} \zeta_{I,J} \ : & Z_{I,J} & \xrightarrow{\cong} & F_{I,J}(Z_{I,J}) \ = \ (J \times Z_{I,J})^I \\ & (\, t : I^\omega \to J^\omega,\ \text{causal}\,) & \longmapsto & \lambda i.\, \big(\, \mathsf{head}(t(i \cdot \vec{i})),\ \lambda \vec{i}'.\, \mathsf{tail}(t(i \cdot \vec{i}'))\,\big) \ . \end{array}$$

Here $i \cdot \vec{i}$ is a letter $i \in I$ followed by an arbitrary stream $\vec{i}$; the value of $\mathsf{head}(t(i \cdot \vec{i}))$ does not depend on $\vec{i}$ since $t$ is causal.

Then there naturally arises a "sequential composition" operation that is different from (2): it acts on *behaviors* of components, simply composing two behaviors of matching types.

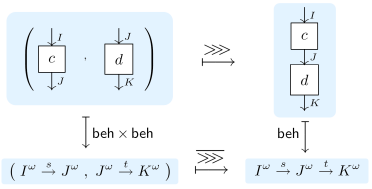
$$\begin{array}{rccccc} \ggg_{I,J,K} \ : & Z_{I,J} & \times & Z_{J,K} & \longrightarrow & Z_{I,K} \\ & \big(\, I^\omega \xrightarrow{s} J^\omega \ , \ J^\omega \xrightarrow{t} K^\omega \,\big) & & & \longmapsto & I^\omega \xrightarrow{s} J^\omega \xrightarrow{t} K^\omega \end{array} \tag{4}$$

The following observation—regarding the two operations (2) and (4)—is crucial for our behavioral view on component calculi. The "inner" operation (4), although it naturally arises by looking at stream functions, is in fact induced by the "outer" operation (2). Specifically, it arises as the behavior map for the (outer) composition $\zeta_{I,J} \ggg_{I,J,K} \zeta_{J,K}$ of two final coalgebras.

$$\begin{array}{ccc} F_{I,K}(Z_{I,J} \times Z_{J,K}) & \dashrightarrow & F_{I,K}(Z_{I,K}) \\ \zeta_{I,J} \ggg \zeta_{J,K}\uparrow & & \text{final}\uparrow\zeta_{I,K} \\ Z_{I,J} \times Z_{J,K} & \xrightarrow[\ggg_{I,J,K}]{} & Z_{I,K} \end{array} \quad \text{i.e.} \quad \left(\begin{array}{c} \downarrow I \\ \boxed{\zeta_{I,J}} \\ \downarrow J \\ \boxed{\zeta_{J,K}} \\ \downarrow K \end{array}\right) \overset{\ggg_{I,J,K}}{\dashrightarrow} \left(\begin{array}{c} \downarrow I \\ \boxed{\zeta_{I,K}} \\ \downarrow K \end{array}\right) \tag{5}$$

Here the coalgebra $\zeta_{I,J} \ggg \zeta_{J,K}$ has a state space $Z_{I,J} \times Z_{J,K}$ due to the definition (3).

As to the two operations (2) and (4), we can ask a further question: are they compatible, in the sense that the the diagram on the right commutes?[9] One can think of this compatibility property as a mathematical formulation of *compositionality*, a fundamental property in the theory of




[8] This is how they are formalized in [28]. Equivalent formulations are: as string functions $I^* \to J^*$ that are length-preserving and prefix-closed [26]; and as functions $I^+ \to J$ where $I^+$ is the set of strings of length $\geq 1$.

[9] The diagram is simplified for brevity. To be precise, the coalgebras $c$ and $d$ must have their states (say $x$ and $y$) specified; these states are mapped to the state $(x, y)$ of $c \ggg d$.

processes/components. The characterization of the inner operation by finality (5) is remarkably useful here; finality immediately yields a positive answer.

In fact, the *microcosm principle* is the mathematical structure that has been behind the story. It refers to the phenomenon that the same algebraic structure is carried by a category $\mathbb{C}$ and by an object $X \in \mathbb{C}$, a prototypical example being *a monoid object in a monoidal category* (see e.g. [24, §VII.3]). In [14] we presented another example eminent in the process theory: parallel composition of two coalgebras for the same signature functor, as well as parallel composition of their behaviors. Our story so far is yet another example—taken from component calculi—with its new feature being that the algebraic structure is many-sorted. In the rest of the paper we develop a categorical language for describing the situation, and proving results about it. Among them is the one that ensures compositionality for a wide class of component calculi.

## 3 FP-theory

### 3.1 Presenting algebraic structure as a category

A component calculus—consisting of operations like $\ggg$ and of equations like associativity of $\ggg$—is an instance of *algebraic specification*. So are (the specifications for) monoids, groups, as well as process calculi such as CCS. Component calculi are different from the other examples here, in that they are *many-sorted*. Such a many-sorted algebraic specification consists of

- a set $\mathcal{S}$ of *sorts*;
- a set $\Sigma$ of *operations*. Each operation $\sigma \in \Sigma$ is equipped with its *in-arity* $\mathsf{inar}(\sigma)$ given by a finite sequence of sorts (denoted as a formal product $S_1 \times \cdots \times S_m$), and its *out-arity* $\mathsf{outar}(\sigma)$ that is some sort $S \in \mathcal{S}$;
- and a set $E$ of equations.

A straightforward presentation of such is simply as a tuple $(\mathcal{S}, \Sigma, E)$ (see e.g. [18]).

In this paper we prefer a different, categorical presentation of algebraic structure. The idea is that an algebraic specification induces a category $\mathbb{L}$ with:

- all the finite sequences of sorts $S_1 \times \cdots \times S_m$ as its objects;
- operations $\sigma \in \Sigma$ as morphisms $\mathsf{inar}(\sigma) \xrightarrow{\sigma} \mathsf{outar}(\sigma)$. Additionally, projections (such as $\pi_1 : S_1 \times S_2 \to S_1$) and diagonals (such as $\langle \mathrm{id}, \mathrm{id} \rangle : S \to S \times S$) are morphisms. So are (formal) products of two morphisms, equipping the category $\mathbb{L}$ with finite products. Besides we can compose morphisms in the category $\mathbb{L}$; that makes the morphisms in $\mathbb{L}$ induced by the *terms* with operations taken from $\Sigma$;
- an equation as a commutative diagram, modulo which we take quotients of terms (as morphisms in $\mathbb{L}$). For example, when $\mathcal{S} = \{*\}$ and $\mathsf{m}$ is a binary operation, its associativity $\mathsf{m}(x, \mathsf{m}(y, z)) = \mathsf{m}(\mathsf{m}(x, y), z)$ amounts to the diagram on the right.

$$\begin{array}{ccc} 3 & \xrightarrow{\mathsf{m}\times\mathrm{id}} & 2 \\ {\scriptstyle \mathrm{id}\times\mathsf{m}}\downarrow & & \downarrow{\scriptstyle \mathsf{m}} \\ 2 & \xrightarrow[\mathsf{m}]{} & 1 \end{array} \qquad (6)$$

In fact, what is represented by $\mathbb{L}$ above is not an algebraic specification itself but its *clone* (see e.g. [10]). In categorical terms, the construction is taking a free finite-product

category from the objects in $\mathcal{S}$ and the arrows in $\Sigma$ modulo the equations induced by $E$. See [18, §3.3] for details.

In a one-sorted setting—where arities (objects of $\mathbb{L}$) are identified with natural numbers by taking their length—such a category $\mathbb{L}$ is called a *Lawvere theory* (see e.g. [14,16,22]). In a many-sorted setting, such a category $\mathbb{L}$—say a "many-sorted Lawvere theory"—is usually called a *finite-product theory*, or an *FP-theory*, see e.g. [4, 5].

**Definition 3.1 (FP-theory)** An *FP-theory* is a category with finite products.

The idea of such categorical presentation of algebraic structure originated in [22]. Significant about the approach is that one has a model as a functor.

**Definition 3.2 (Set-theoretic model)** Let $\mathbb{L}$ be an FP-theory. A *(set-theoretic) model* of $\mathbb{L}$ is a finite-product-preserving (*FP-preserving*) functor $X : \mathbb{L} \to \mathbf{Set}$ into the category $\mathbf{Set}$ of sets and functions.

Later in Def. 4.1 we introduce the notion of *category* with $\mathbb{L}$-structure—this is the kind of models of our interest—based on this standard definition.

To illustrate Def. 3.2 in a one-sorted setting, think about an operation $2 \xrightarrow{\mathsf{m}} 1$ which satisfies associativity (6). Let the image $X(1)$ of $1 \in \mathbb{L}$ be simply denoted by $X$; then $2 = 1 \times 1 \in \mathbb{L}$ must be mapped to the set $X^2$ by FP-preservation. By functoriality the morphism $\mathsf{m}$ is mapped to a morphism $X(\mathsf{m}) : X^2 \to X$ in $\mathbf{Set}$, which we denote by $[\![\mathsf{m}]\!]_X$. This yields a binary operation on the set $X$. Moreover, the associativity diagram (6) in $\mathbb{L}$ is carried to a commutative diagram in $\mathbf{Set}$; this expresses associativity of the interpretation $[\![\mathsf{m}]\!]_X$.

When $\mathbb{L}$ arises from a many-sorted algebraic specification, it is not a single set $X$ that carries $\mathbb{L}$-structure; we have a family of sets $\{X(S)\}_{S \in \mathcal{S}}$—one for each sort $S$—as a carrier. By FP-preservation this extends to interpretation of products of sorts: $X(S_1 \times \cdots \times S_m) \cong X(S_1) \times \cdots \times X(S_m)$. In this way an operation is interpreted with its desired domain and codomain.

**Remark 3.3** *Lawvere theories* and *monads* are the two major ways to represent algebraic structure in category theory (see [16]). Both allow straightforward extension to the many-sorted setting: the former to FP-theories and the latter to monads on functor categories (mostly presheaf categories). For the purpose of formalizing the microcosm principle we find the former more useful. Its representation is independent from the domain where the structure is interpreted; hence we can speak of its models in two different domains, as is the case with the microcosm principle. In contrast, a monad on a category $\mathbb{C}$ specifies algebraic structure (i.e. Eilenberg-Moore algebras) that is necessarily carried by an object of $\mathbb{C}$.

### 3.2 The FP-theory $\mathbf{PLTh}$

We now present a specific FP-theory; this is our working example. We list its sorts, operations and equations; these altogether induce an FP-theory in the way that we sketched above. We denote the resulting FP-theory by $\mathbf{PLTh}$. Later in §5 we will see that this

FP-theory represents Hughes' notion of *arrow*, without its first operation. One can think of $\mathbf{PLTh}$ as a basic component calculus modeling pipelines (PL for "pipeline").

**Assumption 3.4** Throughout the rest of the paper we fix a base category $\mathbb{B}$ to be a cartesian subcategory (i.e. closed under finite products) of $\mathbf{Set}$. Its object $I \in \mathbb{B}$ is a set that can play a role of an interface. Its morphism $f : I \to J$—it is a set-theoretic function—represents a "stateless" computation from $I$ to $J$ that can be realized by a component with a single state.

- The sorts $\mathcal{S} = \{(I, J) \mid I, J \in \mathbb{B}\}$. Hence an object of $\mathbf{PLTh}$ can be written as a formal product $(I_1, J_1) \times \cdots \times (I_m, J_m)$. We denote the nullary product (i.e. the terminal object) by $1 \in \mathbf{PLTh}$;
- the operations:

$$\begin{array}{ll} \ggg_{I,J,K} \ : \ (I, J) \times (J, K) \longrightarrow (I, K) & \textit{sequential composition} \\ \mathsf{arr}\, f \ : \ 1 \longrightarrow (I, J) & \textit{pure function} \end{array}$$

  for each object $I, J, K \in \mathbb{B}$ and each morphism $f : I \to J$ in $\mathbb{B}$. Sequential composition is illustrated in (2). The component $\mathsf{arr}\, f$, intuitively, has a singleton as its state space and realizes "stateless" processing of data stream $\xrightarrow{I}\boxed{\mathsf{arr}\, f}\xrightarrow{J}$;
- the equations:
  - *associativity:*

$$a : (I, J), b : (J, K), c : (K, L) \ \vdash \ a \ggg (b \ggg c) = (a \ggg b) \ggg c \qquad (\ggg\text{-ASSOC})$$

    for each $I, J, K, L \in \mathbb{B}$, omitting the obvious subscripts for $\ggg$, i.e.

$$\begin{array}{ccc} (I, J) \times (J, K) \times (K, L) & \xrightarrow{\ggg_{I,J,K} \times (K,L)} & (I, K) \times (K, L) \\ (I, J) \times \ggg_{J,K,L} \downarrow & & \downarrow \ggg_{I,K,L} \\ (I, J) \times (J, L) & \xrightarrow[\ggg_{I,J,L}]{} & (I, L) \end{array}$$

  - *preservation of composition:* for each composable pair of morphisms $f : I \to J$ and $g : J \to K$ in $\mathbb{B}$,

$$\emptyset \ \vdash \ \mathsf{arr}\,(g \circ f) = \mathsf{arr}\, f \ggg \mathsf{arr}\, g \qquad (\mathsf{arr}\text{-FUNC1})$$

    where $\emptyset$ denotes the empty context. On the right is the corresponding diagram.

$$\begin{array}{ccc} 1 & \xrightarrow{\mathsf{arr}\, f \times \mathsf{arr}\, g} & (I, J) \times (J, K) \\ & \searrow^{\mathsf{arr}\,(g \circ f)} & \downarrow \ggg_{I,J,K} \\ & & (I, K) \end{array}$$

  - *preservation of identities:* for each $I, J \in \mathbb{B}$,

$$a : (I, J) \ \vdash \ \mathsf{arr}\, \mathrm{id}_I \ggg_{I,I,J} a = a = a \ggg_{I,J,J} \mathsf{arr}\, \mathrm{id}_J \ . \qquad (\mathsf{arr}\text{-FUNC2})$$

For this FP-theory, the model of our interest is carried by a family of *categories*, namely the category $\mathbf{Coalg}(F_{I,J})$ for each sort $(I, J)$. Formalization of such an *outer model* carried by categories, together with that of an *inner model* carried by final coalgebras, is the main topic of the next section.

## 4 Microcosm model of an FP-theory

In this section we present our formalization of *microcosm models* for an FP-theory $\mathbb{L}$. It is about nested models of $\mathbb{L}$: the outer one ($\mathbb{L}$-*category*) carried by a family $\{\mathbb{C}(S)\}_{S\in\mathcal{S}}$ of categories; the inner one ($\mathbb{L}$-*object*) carried by a family $\{X_S \in \mathbb{C}(S)\}_{S\in\mathcal{S}}$ of objects. We shall use the formalization to prove a compositionality result (Thm. 4.6).

In fact our formalization is essentially the one in our previous work [14]. Due to the space limit we cannot afford sufficient illustration of our seemingly complicated 2-categorical definitions. The reader is suggested to have [14, §3] as her companion; the thesis [13, Chap. 5] of one of the authors has a more detailed account. What is new here, compared to [14], is the following.

- The algebraic structure of our interest is now many-sorted, generalizing $\mathbb{L}$ from a Lawvere theory to an FP-theory.
- Now our framework accommodates categories with "pseudo" algebraic structure, such as (not strict) monoidal categories. We cannot avoid this issue in the current paper, because the concrete model that we deal with is indeed of such a kind with pseudo algebraic structure.

We will depend heavily on 2-categorical notions such as pseudo functors, lax natural transformations, etc. For these notions the reader is referred to [9].

### 4.1 Outer model: $\mathbb{L}$-category

Take the functor $F_{I,J} = (T(J \times \_))^I$, for which coalgebras are components (see §1). We would like that the categories $\{\mathbf{Coalg}(F_{I,J})\}_{I,J\in\mathbb{B}}$ model $\mathbf{PLTh}$, the algebraic structure for pipelines in §3.2. That is, we need functors

$$\begin{array}{ll} [\![ \ggg_{I,J,K} ]\!] & : \ \mathbf{Coalg}(F_{I,J}) \times \mathbf{Coalg}(F_{J,K}) \to \mathbf{Coalg}(F_{I,K}) \quad \text{for each } I, J, K \in \mathbb{B}, \\ [\![ \mathsf{arr}\, f ]\!] & : \ \mathbf{1} \to \mathbf{Coalg}(F_{I,J}) \qquad \text{for each morphism } f : I \to J \text{ in } \mathbb{B}, \end{array}$$

where $\mathbf{1}$ is a (chosen) terminal category. Moreover these functors must satisfy the three classes of equations of $\mathbf{PLTh}$ in §3.2. One gets pretty close to the desired definition of such "category with $\mathbb{L}$-structure" by replacing "sets" by "categories" in Def. 3.2. That is, by having $\mathbf{CAT}$—the 2-category of locally small categories, functors and natural transformations—in place of $\mathbf{Set}$. In fact we did so in our previous work [14].

However in our coalgebraic modeling of components we want equations to be satisfied not up-to identity, but up-to (coherent) isomorphisms. For example, consider the functor $[\![ \ggg_{I,J,K} ]\!]$ above, in particular how it acts on the carrier sets of coalgebras. By the definition in §2 it carries $(X, Y)$ to $X \times Y$; and this operation is associative only up-to isomorphism. This requirement of "satisfaction up-to iso" also applies to monoidal categories; below is how associativity (on the left) is expected to be interpreted, in a monoid and in a monoidal category.

$$\begin{array}{lll} \underline{\text{in } \mathbb{L}} & \underline{\text{in } \mathbf{Set}} & \underline{\text{in } \mathbf{CAT}} \\ \begin{array}{ccc} 3 & \xrightarrow{\mathsf{id}\times\mathsf{m}} & 2 \\ {\scriptstyle \mathsf{m}\times\mathsf{id}}\downarrow & \swarrow\!\!\swarrow & \downarrow{\scriptstyle \mathsf{m}} \\ 2 & \xrightarrow[\mathsf{m}]{} & 1 \end{array} & \begin{array}{ccc} X^3 & \xrightarrow{[\![\mathsf{id}\times\mathsf{m}]\!]_X} & X^2 \\ {\scriptstyle [\![\mathsf{m}\times\mathsf{id}]\!]_X}\downarrow & \swarrow\!\!\swarrow & \downarrow{\scriptstyle [\![\mathsf{m}]\!]_X} \\ X^2 & \xrightarrow[[\![\mathsf{m}]\!]_X]{} & X \end{array} & \begin{array}{ccc} \mathbb{C}^3 & \xrightarrow{[\![\mathsf{id}\times\mathsf{m}]\!]_{\mathbb{C}}} & \mathbb{C}^2 \\ {\scriptstyle [\![\mathsf{m}\times\mathsf{id}]\!]_{\mathbb{C}}}\downarrow & \overset{\cong}{\swarrow\!\!\swarrow} & \downarrow{\scriptstyle [\![\mathsf{m}]\!]_{\mathbb{C}}} \\ \mathbb{C}^2 & \xrightarrow[[\![\mathsf{m}]\!]_{\mathbb{C}}]{} & \mathbb{C} \end{array} \\ \hline x_1 \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot x_3 & & X_1 \otimes (X_2 \otimes X_3) \xrightarrow{\cong} (X_1 \otimes X_2) \otimes X_3 \end{array} \tag{7}$$

Our approach to such "pseudo algebras" is to relax functorial semantics (Def. 3.2) into *pseudo functorial semantics*, i.e. replacing functor by *pseudo functor*. The idea has been previously mentioned in [14, §3.3] and [13, §5.3.3]; now it has been made rigorous.

**Definition 4.1 ($\mathbb{L}$-category)** An $\mathbb{L}$-*category* is a pseudo functor $\mathbb{C} : \mathbb{L} \to \mathbf{CAT}$ which is "FP-preserving" in the following sense:[10]

1. the canonical map $\langle \mathbb{C}\pi_1, \mathbb{C}\pi_2 \rangle : \mathbb{C}(A_1 \times A_2) \to \mathbb{C}(A_1) \times \mathbb{C}(A_2)$ is an isomorphism for each $A_1, A_2 \in \mathbb{L}$;
2. the canonical map $\mathbb{C}(1) \to \mathbf{1}$ is an isomorphism;
3. it preserves identities up-to identity: $\mathbb{C}(\mathrm{id}) = \mathrm{id}$;
4. it preserves pre- and post-composition of identities up-to identity: $\mathbb{C}(\mathrm{id} \circ \mathsf{a}) = \mathbb{C}(a) = \mathbb{C}(\mathsf{a} \circ \mathrm{id})$;
5. it preserves composition of the form $\pi_i \circ \mathsf{a}$ up-to identity: $\mathbb{C}(\pi_i \circ \mathsf{a}) = \mathbb{C}(\pi_i) \circ \mathbb{C}(\mathsf{a})$. Here $\pi_i : A_1 \times A_2 \to A_i$ is a projection.

We shall often denote $\mathbb{C}$'s action $\mathbb{C}(\mathsf{a})$ on a morphism $\mathsf{a}$ by $[\![\mathsf{a}]\!]_{\mathbb{C}}$.

In the definition, what it means exactly to be "FP-preserving" is a delicate issue; for the current purpose of representing pseudo algebras, we found the conditions above to be the right ones. It is justified by the following result.

**Proposition 4.2** *Let us denote by* $\mathbf{MonTh}$ *the Lawvere theory for monoids. The 2-category* $\mathbf{MonCAT}$ *of monoidal categories, strong monoidal functors and monoidal transformations is equivalent to the 2-category of* $\mathbf{MonTh}$*-categories with suitable 1- and 2-cells.*

*Proof.* The proof involves overwhelming details. It is deferred to [12], where a general result—not only for the specification for monoids but for any algebraic specification—is proved. □

What the last proposition asserts is stronger than merely establishing a *biequivalence* between the two 2-categories, a claim one would expect from e.g. the coherence result for monoidal categories. See [12] for more details about Def. 4.1 and Prop. 4.2; and see [13, §5.3.3] how pseudo functoriality yields the mediating iso 2-cell in (7).

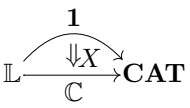
**Remark 4.3** There have been different approaches to formalization of "pseudo algebra." A traditional one (e.g. in [7]) is to find a suitable 2-monad which already takes pseudo satisfaction of equations into account. Another one is by a Lawvere 2-theory, which also includes explicitly the isomorphism up-to which equations are satisfied (see [21]). Neither of them looks quite suitable for the microcosm principle: we want a single representation of algebraic structure interpreted twice; with only the outer one satisfying equations up-to isomorphisms.

[10] To be precise, each of the conditions 3–5 means that the corresponding mediating isomorphism (as part of the definition of a pseudo functor) is actually the identity.

The same idea as ours has been pursued by a few other authors. Segal [29] defines pseudo algebras as pseudo functors, for *monoidal* theories (as opposed to *cartesian* theories in our case), with applications to conformal field theory. Fiore's definition [11] is equivalent to ours, but its aspect as a pseudo functor is not emphasized there.

### 4.2 Inner model: $\mathbb{L}$-object

Once we have an outer model $\mathbb{C}$ of $\mathbb{L}$, we can define the notion of *inner model in* $\mathbb{C}$. It is a family of objects $\{X_S \in \mathbb{C}(S)\}_{S\in\mathcal{S}}$ which carries $\mathbb{L}$-structure in the same way as a monoid in a monoidal category does [24, §VII.3]. We have seen in §2 that final coalgebras carry such an inner model and realize composition of behaviors.

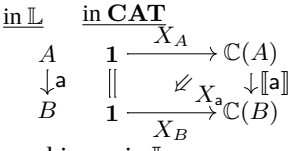



**Definition 4.4 ($\mathbb{L}$-object)** Let $\mathbb{C} : \mathbb{L} \to \mathbf{CAT}$ be an $\mathbb{L}$-category. An $\mathbb{L}$-*object* in $\mathbb{C}$ is a lax natural transformation $X$ as in the diagram above, which is "FP-preserving" in the sense that: it is strictly natural with regard to projections (see [14, Def. 3.4]). Here $\mathbf{1} : \mathbb{L} \to \mathbf{CAT}$ denotes the constant functor into a (chosen) terminal category $\mathbf{1}$.

An $\mathbb{L}$-object is also called a *microcosm model* of $\mathbb{L}$, emphasizing that it is a model that resides in another model $\mathbb{C}$.

The definition is abstract and it is hard to grasp how it works at a glance. While the reader is referred to [13, 14] for its illustration, we shall point out its highlights.

An $\mathbb{L}$-object $X$, as a lax natural transformation, consists of the following data:




- its components $X_A : \mathbf{1} \to \mathbb{C}(A)$, identified with objects $X_A \in \mathbb{C}(A)$, for each $A \in \mathbb{L}$;
- mediating 2-cells $X_{\mathsf{a}}$, as shown on the right, for each morphism $\mathsf{a}$ in $\mathbb{L}$.

Generalizing the illustration in [13, 14] one immediately sees that

- $X$'s components are determined by those $\{X_S\}_{S\in\mathcal{S}}$ on sorts. They extend to an object $S_1 \times \cdots \times S_m$ by:

$$\mathbb{C}(S_1 \times \cdots \times S_m) \ni X_{S_1\times\cdots\times S_m} \xmapsto{\cong} (X_{S_1}, \ldots, X_{S_m}) \in \mathbb{C}(S_1) \times \cdots \times \mathbb{C}(S_m) \ ;$$

- an operation $\sigma$ is interpreted on $X$ by means of the mediating 2-cell $X_\sigma$;
- equations hold due to the coherence condition on the mediating 2-cells: $X_{\mathsf{b}\circ\mathsf{a}}$ must be given by a suitable composition of $X_{\mathsf{a}}$ and $X_{\mathsf{b}}$. In [14, Expl. 3.5] we demonstrate how this coherence condition derives associativity of multiplication for a monoid object (in a monoidal category).

### 4.3 Categorical compositionality

Here we shall present a main technical result, namely the compositionality theorem (Thm. 4.6). It is a straightforward many-sorted adaptation of [14, Thm. 3.9]; to which refer for more illustration of the result.

**Definition 4.5 ($\mathbb{L}$-functor)** Let $\mathbb{C}, \mathbb{D}$ be $\mathbb{L}$-categories. A *lax $\mathbb{L}$-functor* $F : \mathbb{C} \to \mathbb{D}$ is a lax natural transformation $F : \mathbb{C} \Rightarrow \mathbb{D} : \mathbb{L} \to \mathbf{CAT}$ that is FP-preserving in the same sense as in Def. 4.4. Similarly, a *strict $\mathbb{L}$-functor* is a strict natural transformation of the same type.

A lax/strict $\mathbb{L}$-functor determines, as its components, a family of functors $\{F_A : \mathbb{C}(A) \to \mathbb{D}(A)\}_{A \in \mathbb{L}}$. Much like the case for an $\mathbb{L}$-object, it is determined by the components $\{F_S : \mathbb{C}(S) \to \mathbb{D}(S)\}_{S \in \mathcal{S}}$ on sorts.

**Theorem 4.6 (Compositionality)** *Let $\mathbb{C}$ be an $\mathbb{L}$-category, and $F : \mathbb{C} \to \mathbb{C}$ be a lax $\mathbb{L}$-functor. Assume that there is a final coalgebra $\zeta_A : Z_A \xrightarrow{\cong} F_A(Z_A)$ for each $A \in \mathbb{L}$.*

1. *The family $\{\mathbf{Coalg}(F_A)\}_{A \in \mathbb{L}}$ carries an $\mathbb{L}$-category.*
2. *The family $\{\zeta_A \in \mathbf{Coalg}(F_A)\}_{A \in \mathbb{L}}$ carries a microcosm model of $\mathbb{L}$.*
3. *The family $\{\mathbb{C}(A)/Z_A\}_{A \in \mathbb{L}}$ of slice categories carries an $\mathbb{L}$-category.*
4. *The family of functors $\{\mathsf{beh}_A : \mathbf{Coalg}(F_A) \to \mathbb{C}(A)/Z_A\}_{A \in \mathbb{L}}$, where $\mathsf{beh}_A$ is defined by coinduction (see on the right), carries a strict $\mathbb{L}$-functor.*

$$\begin{array}{ccc} F_A X & \dashrightarrow & F_A(Z_A) \\ {\scriptstyle c}\uparrow & & {\scriptstyle final}\uparrow\cong \\ X & \underset{\mathsf{beh}_A(c)}{\dashrightarrow} & Z_A \end{array}$$

*Proof.* The proof is an adaptation of that of [14, Thm. 3.9]; however it involves additional coherence requirements due to the relaxed definition of $\mathbb{L}$-categories. A detailed proof is found in [12]. □

An informal reading of the theorem is as follows. To get a "nice" interpretation of a component calculus $\mathbb{L}$ by $F$-coalgebras, it suffices to check that

- the base category $\mathbb{C}$ models $\mathbb{L}$, and
- the functor $F$ is "lax-compatible" with $\mathbb{L}$.

These data interpret $\mathbb{L}$ on the category of coalgebras, yielding composition of components (Point 1.). Final coalgebras acquire canonical inner $\mathbb{L}$-structure, yielding composition of behaviors (2.). Finally, relating the two interpretations, compositionality is guaranteed (4.).

## 5 Taxonomy of FP-theories for component calculi

Up to now we have kept an FP-theory $\mathbb{L}$ as a parameter and have developed a uniform framework. Now we turn to: concrete models (components as coalgebras); and three concrete FP-theories $\mathbf{PLTh}$, $\mathbf{ArrTh}$ and $\mathbf{MArrTh}$ that express basic component calculi. The latter two are equipped with different "parallel composition" operations.

Notably the algebraic structure expressed by $\mathbf{ArrTh}$ is that of (Hughes') *arrow* [15], equivalently that of *Freyd categories* [23], the notions introduced for modeling structured computations in functional programming.

The main result in this section is that the categories $\{\mathbf{Coalg}(F_{I,J})\}_{I,J}$—modeling components—carry $\mathbf{ArrTh}$-structure. If additionally the effect monad $T$ is commutative, then $\{\mathbf{Coalg}(F_{I,J})\}_{I,J}$ a forteriori carries the stronger $\mathbf{MArrTh}$-structure. These results parallel classic results in categorical semantics of functional programming, investigating (pre)monoidal structure of a Kleisli category.

### 5.1 The FP-theories $\mathbf{ArrTh}$, $\mathbf{MArrTh}$

We shall add, to the FP-theory $\mathbf{PLTh}$ in §3.2, a suitable "parallel composition" operation and equational axioms to obtain the FP-theory $\mathbf{ArrTh}$. By imposing stronger equational axioms we get the FP-theory $\mathbf{MArrTh}$.

In $\mathbf{ArrTh}$ one has additional *sideline* operations

$$\mathsf{first}_{I,J,K} \;:\; (I,J) \longrightarrow (I \times K, J \times K)\;, \quad \text{graphically} \quad \begin{array}{c}\downarrow I\\ \boxed{a}\\ \downarrow J\end{array} \;\overset{\mathsf{first}_{I,J,K}}{\longmapsto}\; \left(\begin{array}{c|c}\downarrow I & K\\ \boxed{a} & \\ \downarrow J & \downarrow K\end{array}\right)$$

for each $I, J, K \in \mathbb{B}$. The equations regarding these are:

$$\begin{aligned}
\mathsf{first}_{I,J,1}\, a \ggg \mathsf{arr}\,\pi &= \mathsf{arr}\,\pi \ggg a && (\rho\text{-NAT})\\
\mathsf{first}_{I,J,K}\, a \ggg \mathsf{arr}(\mathrm{id}_J \times f) &= \mathsf{arr}(\mathrm{id}_I \times f) \ggg \mathsf{first}_{I,J,L}\, a && (\mathsf{arr}\text{-CENTR})\\
(\mathsf{first}_{I,J,K\times L}\, a) \ggg (\mathsf{arr}\,\alpha_{J,K,L}) &= (\mathsf{arr}\,\alpha_{I,K,L}) \ggg \mathsf{first}(\mathsf{first}\, a) && (\alpha\text{-NAT})\\
\mathsf{first}_{I,J,K}(\mathsf{arr}\, f) &= \mathsf{arr}(f \times \mathrm{id}_K) && (\mathsf{arr}\text{-PREMON})\\
\mathsf{first}_{I,K,L}(a \ggg b) &= (\mathsf{first}_{I,J,L}\, a) \ggg (\mathsf{first}_{J,K,L}\, b) && (\mathsf{first}\text{-FUNC})
\end{aligned}$$

In the equations, $f$ denotes a morphism in the base category $\mathbb{B}$. In ($\rho$-NAT), the $\pi$ on the left is the projection $\pi : J \times 1 \overset{\cong}{\to} J$ in $\mathbb{B}$. In ($\alpha$-NAT), $\alpha$'s are associativity isomorphisms like $I \times (J \times K) \overset{\cong}{\to} (I \times J) \times K$ in $\mathbb{B}$.

**Remark 5.1** In fact the equation ($\rho$-NAT) can be derived for any projection $\pi : J \times K \to J$ without requiring $K = 1$. However, the special case above has a clearer role in the corresponding premonoidal structure (see §5.2). Namely, it is the naturality requirement of the right-unit isomorphism $\rho_J = \mathsf{arr}\,\pi_J$ with $\pi_J : J \times 1 \overset{\cong}{\to} J$.

In $\mathbf{MArrTh}$, instead of the operations first, one has

$$\|_{I,J,K,L} :\; (I,J) \times (K,L) \longrightarrow (I \times K, J \times L) \qquad \textit{synchronous composition}$$

for each $I, J, K, L \in \mathbb{B}$. The equations are:

$$\begin{aligned}
(a \parallel b) \ggg (c \parallel d) &= (a \ggg c) \parallel (b \ggg d) && (\|\text{-FUNC1})\\
\mathsf{arr}\,\mathrm{id}_I \parallel \mathsf{arr}\,\mathrm{id}_J &= \mathsf{arr}\,\mathrm{id}_{I\times J} && (\|\text{-FUNC2})\\
a \parallel (b \parallel c) \ggg \mathsf{arr}\,\alpha &= \mathsf{arr}\,\alpha \ggg (a \parallel b) \parallel c && (\alpha\text{-NAT})\\
(a \parallel \mathsf{arr}\,\mathrm{id}_1) \ggg \mathsf{arr}\,\pi &= \mathsf{arr}\,\pi \ggg a && (\rho\text{-NAT})\\
\mathsf{arr}(f \times g) &= \mathsf{arr}\, f \parallel \mathsf{arr}\, g && (\mathsf{arr}\text{-MON})\\
(a \parallel b) \ggg \mathsf{arr}\,\gamma &= \mathsf{arr}\,\gamma \ggg (b \parallel a) && (\gamma\text{-NAT})
\end{aligned}$$

Here $\alpha$'s are associativity isomorphisms, and $\pi$'s are projections like $J \times 1 \overset{\cong}{\to} J$, as in $\mathbf{ArrTh}$. The morphisms $\gamma$ in ($\gamma$-NAT) are symmetry isomorphisms like $J \times I \overset{\cong}{\to} I \times J$ in $\mathbb{B}$. One readily derives, from ($\rho$-NAT) and ($\gamma$-NAT), the equation

$$(\mathsf{arr}\,\mathrm{id}_1 \parallel a) \ggg \mathsf{arr}\,\pi' = \mathsf{arr}\,\pi' \ggg a \qquad (\lambda\text{-NAT})$$

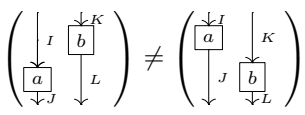
where $\pi'$'s are (second) projections like $1 \times I \overset{\cong}{\to} I$.

The theory $\mathbf{MArrTh}$ is stronger than $\mathbf{ArrTh}$. Indeed, the first operator in $\mathbf{ArrTh}$ can be defined in $\mathbf{MArrTh}$, as: $\mathsf{first}_{I,J,K}\, a := a \parallel \mathrm{id}_K$.

The reason that we have the first operation in $\mathbf{ArrTh}$, instead of $\|$ as in $\mathbf{MArrTh}$, should be noted. The first operation in $\mathbf{ArrTh}$ yields the operation

$$\mathsf{second}_{I,J,K} \;:\; (I,J) \longrightarrow (K \times I, K \times J) \quad \text{by } \mathsf{second}\, a = \mathsf{arr}\,\gamma \ggg \mathsf{first}\, a \ggg \mathsf{arr}\,\gamma\;,$$

where $\gamma$'s are symmetry isomorphisms. But the equations in $\mathbf{ArrTh}$ do not derive $\mathsf{second}\, b \ggg \mathsf{first}\, a = \mathsf{first}\, a \ggg \mathsf{second}\, b$; that is, the two systems on the right should not be identified. There are indeed many situations where they are distinct. Assume that we have the global state monad $T = (S \times \_)^S$ as effect in $F_{I,J}$. One can think of a global state $s \in S$ as residing in the ambience of components, unlike local/internal states that are inside components (i.e. coalgebras). When a component executes, it changes the current global state as well as its internal state; hence the order of execution of $a$ and $b$ does matter. In contrast, when one interprets $\mathbf{MArrTh}$ in $\{\mathbf{Coalg}(F_{I,J})\}_{I,J}$, the natural axiom ($\|$-FUNC1) requires the two systems above to be equal.




### 5.2 Set-theoretic models: arrows, Freyd categories

The FP-theories $\mathbf{PLTh}, \mathbf{ArrTh}$ and $\mathbf{MArrTh}$ and their set-theoretic models are closely related with some notions in semantics of functional programming. Here we elaborate on the relationship; it will be used for calculations later in §5.3.

To start with, $\mathbf{ArrTh}$ is almost exactly the axiomatization of Hughes' *arrow* [15] (specifically the one in [20]), with the only gap explained in Rem. 5.1. The notion of arrow generalizes that of *monad* (modeling effects, i.e. structured output [25, 31]) and that of *comonad* (modeling structured input [30]); the notion of arrow models "structured computations" in general. See e.g. [20, §2.3].

**Definition 5.2** An *arrow* is a set-theoretic model (Def. 3.2) of $\mathbf{ArrTh}$.

It had been folklore, and was proved in [20], that an arrow is the same thing as a *Freyd category* [23, 27]. A Freyd category is a symmetric premonoidal category $\mathbb{K}$ together with a cartesian category $\mathbb{B}$ embedded via an identity-on-object strict premonoidal functor $\mathbb{B} \to \mathbb{K}$, subject to a condition on center morphisms. In this sense $\mathbf{ArrTh}$ is an axiomatization of Freyd categories. There is a similar corresponding structure for the stronger FP-theory $\mathbf{MArrTh}$, which derives its name (M for *monoidal*).

**Definition 5.3** Let $\mathbb{B}$ be a cartesian (hence monoidal) category. A *monoidal Freyd category* on $\mathbb{B}$ is a symmetric monoidal category $\mathbb{K}$ together with a strictly monoidal, identity-on-object functor $\mathbb{B} \to \mathbb{K}$.

**Proposition 5.4** *1. A set-theoretic model of* $\mathbf{ArrTh}$ *is the same as a Freyd category.*
*2. A set-theoretic model of* $\mathbf{MArrTh}$ *is the same thing as a monoidal Freyd category.*

*Proof.* The first point is simply the correspondence result [20, Thm. 6.1] restated, using Def. 4.4. The proof of the second point goes similar. □

The notions of (monoidal) Freyd category were introduced in [27], prior to arrows, as axiomatizations of the structure possessed by a Kleisli category $\mathcal{K}\ell(T)$ for a monad $T$. Here a Kleisli category is the category of types and effectful computations [25]. The results [27, Cor. 4.2 & 4.3]—showing that $\mathcal{K}\ell(T)$ indeed possesses such structure—now read as follows. For the notions of strong/commutative monad, see e.g. [17, §3].

**Proposition 5.5** *Let $\mathbb{B}$ be our base category (see Assumption 3.4).*

1. *A monad $T$ on $\mathbf{Set}$ induces a model $\mathcal{K}\ell(T) : \mathbf{ArrTh} \to \mathbf{Set}$. Specifically, its carrier set $\mathcal{K}\ell(T)(I,J)$ for a sort $(I,J)$ is the homset $\mathrm{Hom}_{\mathcal{K}\ell(T)}(I,J)$; $\mathsf{arr}$ is interpreted by the Kleisli inclusion functor; $\ggg$ is by composition in $\mathcal{K}\ell(T)$; and $\mathsf{first}$ is obtained using $T$'s canonical strength $\mathsf{st}$. Note that every monad on $\mathbf{Set}$ is strong.*
2. *Furthermore, if $T$ is commutative then it induces a model $\mathcal{K}\ell(T)$ of $\mathbf{MArrTh}$. The operation $\|$ is interpreted using $T$'s double strength.* □

Thanks to the proposition we know that all the equations in $\mathbf{ArrTh}$ or in $\mathbf{MArrTh}$ hold in $\mathcal{K}\ell(T)$. This will be heavily exploited in the equational reasoning later in §5.3.

For $\mathbf{PLTh}$—the parallel-free part of $\mathbf{ArrTh}$ and $\mathbf{MArrTh}$—a set-theoretic model is an arrow without first, or equivalently, a category $\mathbb{K}$ with an identity-on-object functor $\mathbb{B} \to \mathbb{K}$. Yet another characterization of this structure, discovered in [20], is as a monoid object in the monoidal category of bifunctors $[\mathbb{B}^{\mathrm{op}} \times \mathbb{B}, \mathbf{Set}]$. Equivalently it is a monad on $\mathbb{B}$ in the bicategory of profunctors (also called distributors, see e.g. [6, 9]).

### 5.3 PLTh, ArrTh and MArrTh as component calculi

We now show that our coalgebraic modeling of components indeed models the calculi $\mathbf{PLTh}$, $\mathbf{ArrTh}$ and $\mathbf{MArrTh}$, depending on the monad $T$. The result parallels Prop. 5.5. Throughout the rest of the section we denote by $\mathbb{L}_{\mathrm{CC}}$ any one of $\mathbf{PLTh}$, $\mathbf{ArrTh}$ and $\mathbf{MArrTh}$ (CC for "component calculus").

In view of Thm. 4.6, we only need to establish that: 1) the (constant) map $(I,J) \mapsto \mathbf{Set}$ extends to an $\mathbb{L}_{\mathrm{CC}}$-category; and 2) $\{F_{I,J} : \mathbf{Set} \to \mathbf{Set}\}_{I,J}$ extends to a lax $\mathbb{L}_{\mathrm{CC}}$-functor. Then Thm 4.6 ensures that the components (as coalgebras) and their behaviors (as elements of final coalgebras) carry a microcosm model of $\mathbb{L}_{\mathrm{CC}}$, and that compositionality holds between the two levels.

For 1), we interpret the operations in the following way. The guiding question is: what is the state space of the resulting component.

**Definition 5.6** We denote by $\mathbf{Set}$ the $\mathbb{L}_{\mathrm{CC}}$-category defined as follows. It maps each sort $(I,J)$ to $\mathbf{Set} \in \mathbf{CAT}$; and it interprets operations by

$$\begin{array}{cccccccccc} \mathbf{1} & \xrightarrow{[\![\mathsf{arr}\ f]\!]} & \mathbf{Set}, & \mathbf{Set} & \xrightarrow{[\![\mathsf{first}]\!]} & \mathbf{Set}, & \mathbf{Set}\times\mathbf{Set} & \xrightarrow{[\![\ggg]\!]} & \mathbf{Set}, & \mathbf{Set}\times\mathbf{Set} \xrightarrow{[\![\|]\!]} \mathbf{Set}. \\ * & \longmapsto & 1 & X & \longmapsto & X & (X,Y) & \longmapsto & X\times Y & (X,Y) \longmapsto X\times Y \end{array}$$

We shall use the same notation $\mathbf{Set}$ for models of three different FP-theories.

**Lemma 5.7** *The data in Def. 5.6 indeed determine an FP-preserving pseudo functor. In particular, all the equations in $\mathbf{PLTh}$, $\mathbf{ArrTh}$ and $\mathbf{MArrTh}$ are satisfied up-to coherent isomorphisms.* □

The equations hold only up-to isomorphisms because, for example, the associativity $X \times (Y \times U) \cong (X \times Y) \times U$ in $\mathbf{Set}$ is only an isomorphism.

The requirement 2)—that $\{F_{I,J} : \mathbf{Set} \to \mathbf{Set}\}_{I,J\in\mathbb{B}}$ extends to a lax $\mathbb{L}_{\mathrm{CC}}$-functor—puts additional demands on a monad $T$ which appears as a parameter in $F_{I,J} = (T(J \times$

$\_))^I$. This is parallel to Prop. 5.5. Still the actual calculation is overwhelming. We notice that all the operations that appear throughout the calculation can be described as morphisms in the Kleisli category $\mathcal{K}\ell(T)$. Therefore, by Prop. 5.5, they are themselves subject to the equations in $\mathbb{L}_{\mathrm{CC}}$. This substantially simplifies the calculation.

**Lemma 5.8** *For the endofunctors $F_{I,J} = (T(J \times \_))^I$, the following hold.*

1. *The family $\{F_{I,J} : \mathbf{Set} \to \mathbf{Set}\}_{I,J}$ extends to a lax $\mathbf{ArrTh}$-functor $\mathbf{Set} \to \mathbf{Set}$. Therefore so it does to a lax $\mathbf{PLTh}$-functor.*
2. *If $T$ is commutative, $\{F_{I,J} : \mathbf{Set} \to \mathbf{Set}\}_{I,J}$ extends to a lax $\mathbf{MArrTh}$-functor $\mathbf{Set} \to \mathbf{Set}$.*

*Proof.* What we need to do is: first to "interpret operations" on $\{F_{I,J}\}_{I,J}$; second to check if these interpreted operations "satisfy equations."

More specifically, to make $\{F_{I,J}\}_{I,J}$ into a lax $\mathbb{L}_{\mathrm{CC}}$-functor, we need to have a mediating 2-cell corresponding to each operation (such as $\ggg$). For example:

$$\underline{\text{in } \mathbb{L}_{\mathrm{CC}}} \quad \begin{array}{c}(I,J)\times(J,K)\\ \downarrow{\ggg}\\ (I,K)\end{array} \qquad \underline{\text{in } \mathbf{CAT}} \quad \begin{array}{ccc}\mathbf{Set}\times\mathbf{Set} & \xrightarrow{F_{I,J}\times F_{J,K}} & \mathbf{Set}\times\mathbf{Set}\\ [\![\ggg]\!]=\mathbf{\times}\downarrow & \swarrow F_{\ggg} & \downarrow[\![\ggg]\!]=\mathbf{\times}\\ \mathbf{Set} & \xrightarrow[F_{I,K}]{} & \mathbf{Set}\end{array} \tag{8}$$

In the diagram, we have denoted the binary product in $\mathbf{Set}$ by the boldface $\mathbf{\times}$ to distinguish it from the binary product $\times$ in $\mathbf{CAT}$. The needed 2-cell $F_{\ggg}$ is nothing but a natural transformation

$$F_{\ggg} \;:\; F_{I,J}X \times F_{J,K}Y \longrightarrow F_{I,K}(X \times Y) \;. \tag{9}$$

Finding such $F_{\ggg}$ is essentially what we did in the beginning of §2; this time we shall do that for a general monad $T$ (especially involving $T$'s multiplication $\mu$).

After that we have to show that these mediating 2-cells satisfy the coherence condition. In the current setting where the domain category $\mathbb{L}_{\mathrm{CC}}$ is syntactically generated from $(\mathcal{S}, \Sigma, E)$, it amounts to checking if the mediating 2-cells "satisfy the equations." Taking ($\ggg$-ASSOC) as an example, it means that the following 2-cells are equal.

$$\mathbf{\times}\circ(\mathbf{\times}\times\mathrm{id})\left(\begin{array}{ccc}\mathbf{Set}^3 & \xrightarrow{F_{I,J}\times F_{J,K}\times F_{K,L}} & \mathbf{Set}^3\\ {\scriptstyle \mathrm{id}\times\mathbf{\times}}\downarrow & \swarrow_{\mathrm{id}\times F_{\ggg}} & \downarrow{\scriptstyle \mathrm{id}\times\mathbf{\times}}\\ \mathbf{Set}^2 & \xrightarrow{F_{I,K}\times F_{K,L}} & \mathbf{Set}^2\\ {\scriptstyle \mathbf{\times}}\downarrow & \swarrow F_{\ggg} & \downarrow{\scriptstyle \mathbf{\times}}\\ \mathbf{Set} & \xrightarrow[F_{I,L}]{} & \mathbf{Set}\end{array}\right) = \left(\begin{array}{ccc}\mathbf{Set}^3 & \xrightarrow{F_{I,J}\times F_{J,K}\times F_{K,L}} & \mathbf{Set}^3\\ {\scriptstyle \mathbf{\times}\times\mathrm{id}}\downarrow & \swarrow_{F_{\ggg}\times\mathrm{id}} & \downarrow{\scriptstyle \mathbf{\times}\times\mathrm{id}}\\ \mathbf{Set}^2 & \xrightarrow{F_{I,K}\times F_{K,L}} & \mathbf{Set}^2\\ {\scriptstyle \mathbf{\times}}\downarrow & \swarrow F_{\ggg} & \downarrow{\scriptstyle \mathbf{\times}}\\ \mathbf{Set} & \xrightarrow[F_{I,L}]{} & \mathbf{Set}\end{array}\right)\mathbf{\times}\circ(\mathrm{id}\times\mathbf{\times}) \tag{10}$$

Here $\alpha$ denotes the associativity isomorphism. See [13, Rem. 5.4.1] for more illustration. One readily sees that (10) boils down to commutativity of the following diagram.

$$\begin{array}{ccccc}F_{I,J}X \times (F_{J,K}Y \times F_{K,L}U) & \xrightarrow{\mathrm{id}\times F_{\ggg}} & F_{I,J}X \times F_{J,L}(Y \times U) & \xrightarrow{F_{\ggg}} & F_{I,L}(X \times (Y \times U))\\ {\scriptstyle \alpha}\downarrow & & & & \downarrow{\scriptstyle F_{I,L}\alpha}\\ (F_{I,J}X \times F_{J,K}Y) \times F_{K,L}U & \xrightarrow[F_{\ggg}\times\mathrm{id}]{} & F_{I,K}(X \times Y) \times F_{K,L}U & \xrightarrow[F_{\ggg}]{} & F_{I,L}((X \times Y) \times U)\end{array} \tag{11}$$

To summarize: we shall introduce a natural transformation $F_\sigma$, for each operation $\sigma$, like in (9); and check if they satisfy all the equations like in (11). While the first task is straightforward, the second is painfully complicated as it is. We shall only do the aforementioned examples in $\mathbb{L}_{\mathrm{CC}}$ for demonstration.

In the sequel, let us denote the set-theoretic model of $\mathbb{L}_{\mathrm{CC}}$ induced by the Kleisli category (in Prop. 5.5) by $\mathcal{K}\ell(T)$. In particular, we have $\mathcal{K}\ell(T)(I,J) = (TJ)^I$. Hence the natural transformation $F_{\ggg}$ in (9) is of the type

$$\mathcal{K}\ell(T)(I, J \times X) \times \mathcal{K}\ell(T)(J, K \times Y) \to \mathcal{K}\ell(T)(I, K \times (X \times Y)) \ .$$

We define it to be the following composition of morphisms in $\mathbb{L}_{\mathrm{CC}}$, interpreted in $\mathcal{K}\ell(T)$.

$$\begin{array}{c} (I, J \times X) \times (J, K \times Y) \overset{\mathsf{id} \times \mathsf{first}}{\longrightarrow} (I, J \times X) \times \big(J \times X,\, (K \times Y) \times X\big) \\ \overset{\ggg}{\longrightarrow} \big(I,\, (K \times Y) \times X\big) \overset{(\_) \ggg (\mathsf{arr}\, \alpha^{-1})}{\longrightarrow} (I, K \times (X \times Y)) \end{array} \tag{12}$$

One can readily come up with such a morphism in $\mathbb{L}_{\mathrm{CC}}$ for each of the other operations.

Let us prove that $F_{\ggg}$ thus defined satisfies (11). The morphisms in the diagram (11) can be also written as morphisms in $\mathbb{L}_{\mathrm{CC}}$, interpreted in $\mathcal{K}\ell(T)$. Hence we can use the equational axioms of $\mathbb{L}_{\mathrm{CC}}$ (§5.1); in fact they are enough to show commutativity of (11). In the calculation below, we denote the interpretation $[\![\ggg]\!]_{\mathcal{K}\ell(T)}$ simply by $\ggg$; the same for the other operations first and arr. To reduce the number of parentheses, the terms are presented modulo the associativity ($\ggg$-ASSOC).

$$\begin{array}{ll}
\big(F_{\ggg} \circ (F_{\ggg} \times \mathrm{id}) \circ \alpha\big)(c,(d,e)) & \\
= F_{\ggg}(\, F_{\ggg}(c,d),\, e\,) & \\
= c \ggg \mathsf{first}\, d \ggg \mathsf{arr}\, \alpha^{-1} \ggg \mathsf{first}\, e \ggg \mathsf{arr}\, \alpha^{-1} & \\
= c \ggg \mathsf{first}\, d \ggg \mathsf{first}\, \mathsf{first}\, e \ggg \mathsf{arr}\, \alpha^{-1} \ggg \mathsf{arr}\, \alpha^{-1} & (\alpha\text{-NAT}) \\
= c \ggg \mathsf{first}\, d \ggg \mathsf{first}\, \mathsf{first}\, e \ggg \mathsf{arr}\, (\alpha^{-1} \circ \alpha^{-1}) & (\mathsf{arr}\text{-FUNC1}) \\
= c \ggg \mathsf{first}\, d \ggg \mathsf{first}\, \mathsf{first}\, e \ggg \mathsf{arr}\, \big((L \times \alpha) \circ \alpha^{-1} \circ (\alpha^{-1} \times U)\big) & (\dagger) \\
= c \ggg \mathsf{first}\, d \ggg \mathsf{first}\, \mathsf{first}\, e \ggg \mathsf{arr}\, (\alpha^{-1} \times U) \ggg \mathsf{arr}\, \alpha^{-1} \ggg \mathsf{arr}\, (L \times \alpha) & (\mathsf{arr}\text{-FUNC1}) \\
= c \ggg \mathsf{first}\, d \ggg \mathsf{first}\, \mathsf{first}\, e \ggg \mathsf{first}\, (\mathsf{arr}\, \alpha^{-1}) \ggg \mathsf{arr}\, \alpha^{-1} \ggg \mathsf{arr}\, (L \times \alpha) & (\mathsf{first}\text{-PREMON}) \\
= c \ggg \mathsf{first}\big((d \ggg \mathsf{first}\, e) \ggg \mathsf{arr}\, \alpha^{-1}\big) \ggg \mathsf{arr}\, \alpha^{-1} \ggg \mathsf{arr}(L \times \alpha) & (\mathsf{first}\text{-FUNC}) \\
= (F\alpha \circ F_{\ggg})\big(c, F_{\ggg}(d,e)\big) & \\
= \big(F\alpha \circ F_{\ggg} \circ (\mathrm{id} \times F_{\ggg})\big)\big)(c,(d,e)) &
\end{array}$$

Here the equality $(\dagger)$ is because of the "pentagon" coherence for $\alpha$ in $\mathbf{Set}$. Naturality of $F_{\ggg}$ in $X, Y$, as well as satisfaction of the other equations, can be derived by similar calculation. □

Let us summarize our approach. In the categorical model $\{\mathbf{Coalg}(F_{I,J})\}$ of $\mathbb{L}_{\mathrm{CC}}$ that we are interested in, operations are interpreted essentially by $\{F_{I,J}\}$'s lax compatibility with $\mathbb{L}_{\mathrm{CC}}$ (Thm. 4.6). Furthermore we notice that the latter can be described using $\mathbb{L}_{\mathrm{CC}}$'s specific set-theoretic model $\mathcal{K}\ell(T)$, which we exploit (the above proof). Note that we can employ $\mathcal{K}\ell(T)$ specifically because of the shape $(T(J \times \_))^I$ of $F_{I,J}$.

For obtaining a microcosm model we need final coalgebras. This depends on the "size" of the monad $T$ (see e.g. [26]); all the examples of $T$ listed in §1, except for probabilistic branching and unbounded non-determinism, satisfy this size requirement.

**Theorem 5.9** *Assume that, for each $I, J \in \mathbb{B}$, we have a final coalgebra $\zeta_{I,J} : Z_{I,J} \xrightarrow{\cong} F_{I,J}(Z_{I,J})$.*

1. *The family $\{\mathbf{Coalg}(F_{I,J})\}_{I,J}$ carries an* $\mathbf{ArrTh}$*-category, so a* $\mathbf{PLTh}$*-category.*
2. *The family $\{Z_{I,J} \in \mathbf{Set}\}_{I,J}$ carries an* $\mathbf{ArrTh}$*-object, hence a* $\mathbf{PLTh}$*-object. Compositionality holds in the sense of Thm. 4.6.4.*
3. *If $T$ is commutative, then the above two are also an* $\mathbf{MArrTh}$*-category and an* $\mathbf{MArrTh}$*-object, respectively.*

*Proof.* By Thm. 4.6, Lem. 5.7 and Lem. 5.8. □

## 6 Conclusions and Future Work

We have extended our previous formalization of the microcosm principle [14] to a many-sorted setting. This allowed to include Barbosa's component calculi [2] as examples. We studied three concrete calculi that are variants of the axiomatization of arrows, demonstrating similarity between components and structured computations.

As future work, we are interested in further extensions of the component calculi that allow modeling of further interesting examples like wiring and merging components, queues, stacks, and folders of stacks with feedback, etc., as presented in [2]. The proof methods that we derived from the microcosm framework will be useful in its course.

On the more abstract side, it is interesting to elevate the arguments in §5 further, to the bicategory $\mathbf{DIST}$ of distributors, with $\mathbf{CAT}$ embedded in it via the Yoneda embedding. Such a higher level view on the matter might reveal further microcosm instances in our proof methods.

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